OXFORD IB DIPLOMA PROGRAMME

IN COOPERATION WITH ib

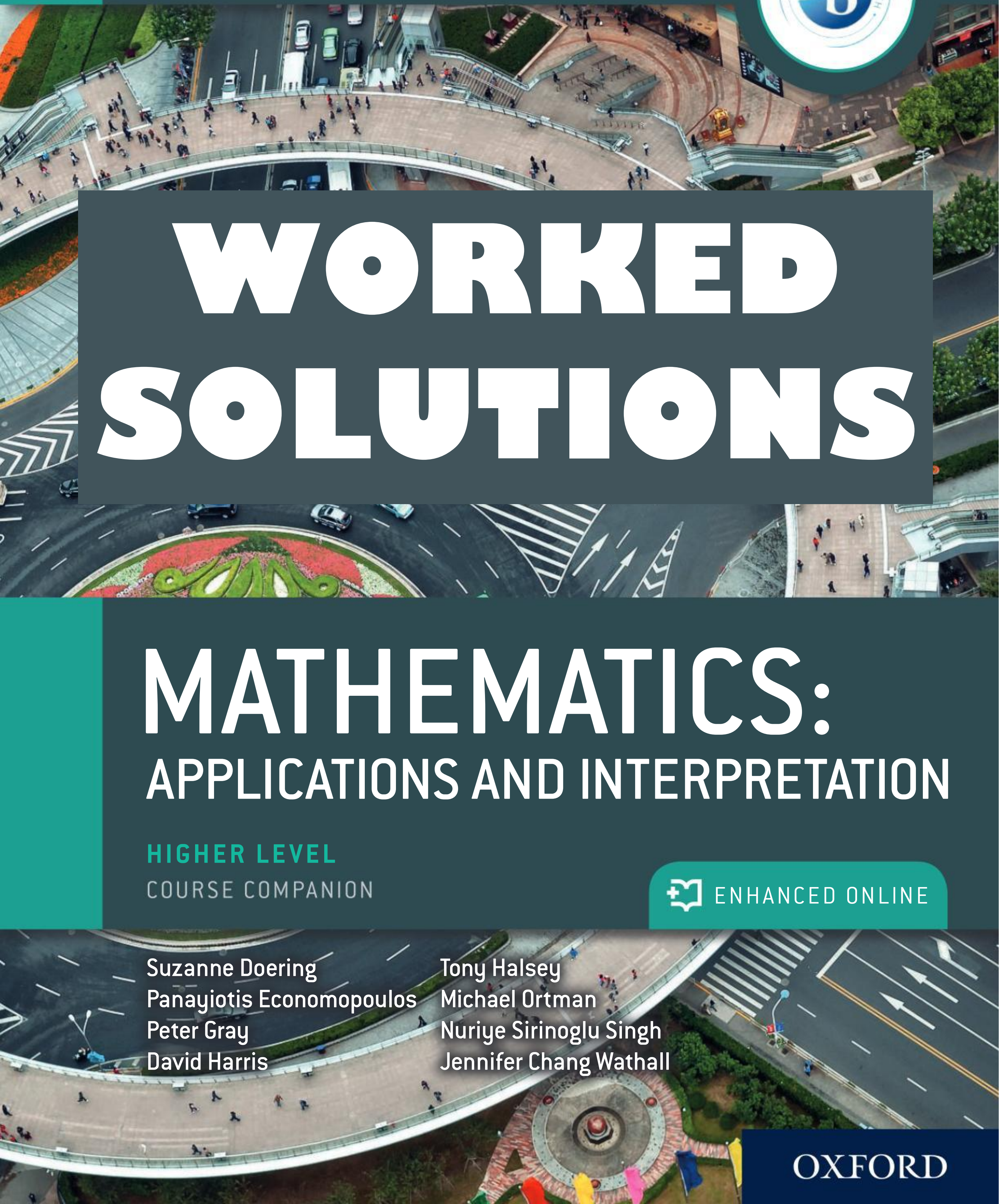
WORKED SOLUTIONS

MATHEMATICS:
APPLICATIONS AND INTERPRETATION

HIGHER LEVEL

COURSE COMPANION

ENHANCED ONLINE



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OXFORD

1 Measuring space: accuracy and geometry

Skills check

1 **a** **i** 0.69 **ii** 28.71 **iii** 77.98

b **i** 0.694 **ii** 28.7 **iii** 78.0.

2 **a** $2^{-3} = \frac{1}{8}$ **b** $27^{\frac{1}{3}} = 3$

3 **a** $x^2 = 9^2 + 13^2 = 250,\ x = 5\sqrt{10}$ **b** $7^2 = x^2 + 5^2,\ x^2 = 24,\ x = 2\sqrt{6}$.

Exercise 1A

1 Circumference = 4.1 = *2πr,* thus *r* = 2.05/π.

Area of circles = $\pi r^2 = \pi\,(2.05/\pi)^2 = 1.3376\ldots$

The side of the length of the new square tables: $l = \sqrt{A} = 1.1565\ldots$, so rounded results are:

i 1.2m (accuracy of the least accurate measurement)

ii 1.16m.

2 Average height of the koalas: $\frac{81+73+71+80+76+84+73+88+91+75}{10} = 79.2$, rounded to 2 s.f. $= 79\,\text{cm}$

Exercise 1B

1 **a** $23.5\text{ mm} \le a < 24.5\text{ mm}$ **b** $3.15\text{ m} \le a < 3.25\text{ m}$

c $1.745\text{ kg} \le a < 1.755\text{ kg}$ **d** $1395\text{g} \le a < 1405\text{g}$

2 **a** Percentage error $= \left|\frac{8840 - 8848}{8848}\right| \times 100\% = 0.09\%\ (2\text{d.p.})$

b $8847.5\,\text{m} \le h < 8848.5\,\text{m}$

3 **a** Average for group 1: $\frac{0.45+0.53+0.47+0.55+0.43+0.67}{6} = 0.517\ (3\text{ s.f.})$, average for group 2: $\frac{0.48+0.56+0.34+0.49+0.30+0.45}{6} = 0.436\ (3\text{ s.f.})$

b Percentage error for group 1 $= \left|\frac{0.5166\ldots - 0.452}{0.452}\right| \times 100\% = 14.3\%\ (3s.f.)$, percentage error for group 2 $= \left|\frac{0.4366\ldots - 0.452}{0.452}\right| \times 100\% = 3.39\%\ (3s.f.)$

c The uncertainty of the results is much larger for the measurements done by group 1.

4 Range of possible values for the number of bicycles: $71.5\,\text{million} \le n < 72.5\,\text{million}$, range of possible values for the average distance travelled: $1.5\,\text{km} \le d < 2.5\,\text{km}$, hence the upper bound for total distance travelled by all the bicycles = $72.5 \times 2.5 \times 365 = 66 \times 10^9$ km (2 s.f.) .

5 $P = 1 - \frac{c}{s}$, so maximum relative profit will be achieved for smallest c and largest s:

$P_{\text{max}} = 1 - \frac{225000}{345000} = 0.348$ (3 s.f.). Similarly, minimum relative profit will be achieved for largest c and smallest s: $P_{\text{min}} = 1 - \frac{235000}{335000} = 0.299$ (3 s.f.).

6 **a** Actual value: $\frac{5}{9}(50 - 32) = 10$ °C, approximate value: $\frac{1}{2}(50 - 32) = 9$ °C.

b Percentage error $= \left|\frac{9 - 10}{10}\right| \times 100\% = 10\%$.

7 **a** The error of the area: $0.002 \times 163 = 0.326\,\text{m}^2$, so the range of values of the area is: $162.674\,\text{m}^2 \le A < 163.326\,\text{m}^2$. Since $A = \pi r^2$, the range of possible values of the radius is then $7.20\,\text{m} \le r < 7.21\,\text{m}$ (3 s.f.)

b The radius must be correct to $u = 0.005\,\text{m}$.

Exercise 1C

1 **a** **i** $(1\times10^{-3})(9\times10^{7}) = 9\times10^{4}$ **ii** 9.936×10^4

b **i** $\frac{7\times10^{4}}{7\times10^{-6}} = 1\times10^{10}$ **ii** 9.669×10^9 (3 d.p.)

2 **a** 9.40×10^{-5} **b** 8.35×10^3 **c** 5.24×10^{-19} **d** 3.87×10^{-7}.

3 **a** **i** $\frac{15}{x^{\frac{1}{2}}}$ **ii** $15x^{-\frac{1}{2}}$

b **i** 7 **ii** 1×7^1

c **i** $\frac{1}{2^{3+3t}}$ **ii** $1 \times 2^{-3-3t}$

d **i** $\frac{25}{3^{2x}}$ **ii** 25×3^{-2x}.

4 **a** $B(1) = 240$, $B\left(\frac{3}{2}\right) = 240 \times \sqrt{2} = 339$ (3 s.f.), $B(2) = 480$.

b The rate of the growth of the bacteria is increasing. When t=3/2, we have the number of bacteria after one and a half hours.

5 **a** $I = \frac{1600}{2^{\frac{t}{8}}}$. **b** $I = \frac{1600}{\sqrt{2}} \approx 1130$ (3 s.f.)

6 The width of the actual speck of dust is $\frac{1.2 \times 10^2}{5 \times 10^2} = 0.24\,\text{mm}$.

7 Earth is $\frac{5.97 \times 10^{24}}{3.29 \times 10^{23}} = 1.81 \times 10^1 = 18.1$ times more massive than Mercury (3 s.f.).

8 Population density of the Earth is $\frac{7.6\times10^9}{5.1\times10^8} = 1.49\times10^1 \approx 15$ people per square kilometer (2 s.f.). Note: values given in the question are to 2 significant figures, so the answers should also be given to 2 significant figures.

Excluding the oceans, the population density is $\frac{7.6\times10^9}{0.3\times5.1\times10^8} = 49.673 \approx 50$ people per square kilometer (2 s.f.).

Exercise 1D

1 **a** $\theta = 90.0 - 61.2 = 28.8^o$, $v = 9\,\text{cm} \times \sin 61.2^o = 8$ cm, $w = 9\,\text{cm}\times\cos 61.2^o = 4\,\text{cm}$.

b $\theta = 90 - 34 = 56^o$, $y = \frac{10.2}{\sin 34^o} = 18.2\,\text{cm}$, $x = \frac{10.2}{\tan 34°} = 15.1\,\text{cm}$.

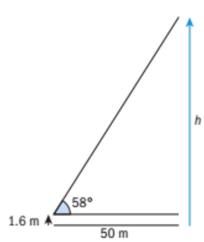
c $z = \sqrt{3.2^2 + 4.7^2} = 5.7\,\text{cm}, \sin\alpha = \frac{4.7}{z} = 0.8$, $\alpha = 56^o, \sin\beta = \frac{3.2}{z} = 0.6$, $\beta = 34^o$.

2 **i** $KL = 8.5\times\cos 30^o = 7.36\,\text{m}$ (3 s.f.)

ii $ML = 8.5\times\sin 30^o = 4.25\,\text{m}$ (3 s.f.)

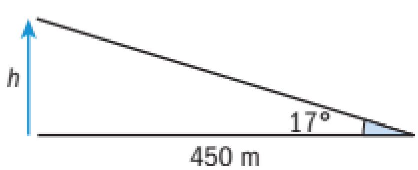
iii $ML_{max} = 8.5\times\sin 55^o = 6.96\,\text{m}$ (3 s.f.)

3 **a**



b Height of the cliff $h = 1.6 + 50\times\tan 58^o = 81.6\,\text{m}$ (3 s.f.)

4 **a**



b Height of the cliff: $h = 450\times\tan 17^o = 138\,\text{m}$.

5 Length of the awning: $l = \frac{2.80}{\tan 70°} = 1.02$ m (3 s.f.)

6 Depth of the crater: $d = 606\times\tan 65^o = 1300\,\text{m}$.

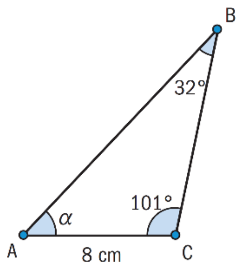
7 If radius of the Earth is $6.4\times10^6\,\text{m}$, use Pythagoras theorem to find the straight line distance between the balcony and the ship: $d = \sqrt{(6.4\times10^6 + 155)^2 - 6.4\times10^6} = 44500\,\text{m} = 44.5$ km (3 s.f.)

8 a Length of the ramp: $l = \dfrac{27}{\sin 13^o} = 120\,\text{cm} = 1.2\ \text{m}$

b The range of possible values of the length: $119.5\,\text{cm} \le l < 120.5\,\text{cm}$. Then, the range of the possible values of the sine of the angle:
$\dfrac{27.43}{120.5} \le \sin\theta < \dfrac{27.43}{119.5}$, $0.2276 \le \sin\theta < 0.2295$, $13.16 \le \theta < 13.27$, so maximal percentage error will be $= \dfrac{13.27 - 13.00}{13.00} \times 100\% = 2.08\%\ (3\ \text{s.f.})$

Exercise 1E

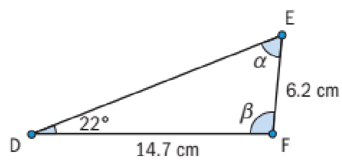
1 a i



ii Only one. The angle is determined from the sum of the angles for a triangle and the missing lengths are determined by the sine rule.

iii $\alpha = 47^o$, $AB = \dfrac{8}{\sin 32^o}\sin 101^o = 14.8\,\text{cm}$, $BC = \dfrac{8}{\sin 32^o}\sin 47^o = 11.0\text{cm}$ (3 s.f.)

b i



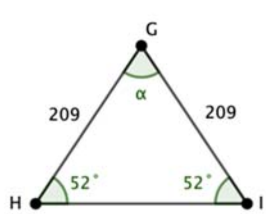
ii Two: the angle α is determined by the sine rule so there are two possibilities, then angle β is determined from the sum of the angles for a triangle, DE is determined by the sine rule.

iii $\sin\alpha = 14.7 \times \dfrac{\sin 22}{6.2}$, $\alpha = 62.6^o$, $\beta = 95.4^o$, $DE = \dfrac{6.2}{\sin 22^o}\sin 95.4^o = 16.5\,\text{cm}$

or, $\alpha = 180 - 62.6 = 117.4^o$, $\beta = 180 - 22 - 117.4 = 40.6^o$, $DE = \dfrac{6.2}{\sin 22^o}\sin 40.6^o = 10.8\,\text{cm}$

(3 s.f.).

c i



ii Only one. The angle α is determined from the sum of angles for a triangle and then the missing length is determined from the sine rule.

iii $\alpha = 180 - 2 \times 52 = 76^o$, $HI = \dfrac{209}{\sin 52}\sin 76^o = 257\,\text{cm}$ (3 s.f.)

2 To find the distance AB, use sine rule: $AB = \sin A\hat{C}B \times \dfrac{AC}{\sin A\hat{B}C} = \dfrac{\sin 55^o \times 450}{\sin(180-55-45)} = 374\,\text{m}$

3 First, find the following angles in triangles QPT and PTS:

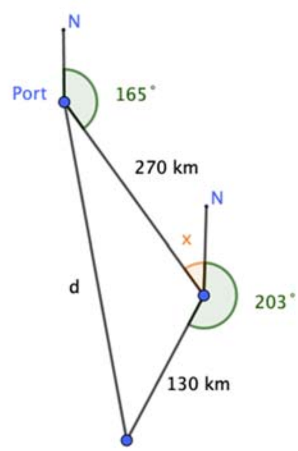
$\hat{P} = 37°, \hat{Q} = 50 - 37 = 13^o, T\hat{P}S = Q\hat{S}P = \dfrac{180-50}{2} = 65^o, Q\hat{T}P = 180 - 13 - 37 = 130°$.

From sine rule applied to triangle QPS: $QP = \dfrac{11}{\sin(65^o + 37^o)} \times \sin 65^o = 10.2\,\text{m}$

Then, apply sine rule to triangle QPT:

$QT = \dfrac{10.2}{\sin 130^o} \times \sin 37^o = 8.01\,\text{m}, PT = 11 - 8.01 = 2.99\ \text{m}$ (3 s.f.)

4



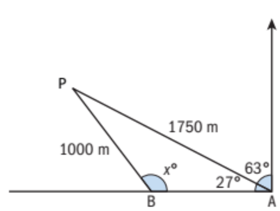
Angle *x=180°-165°=15°*

Angle opposite length d is 360°-203°-15°=142°

Use the cosine rule to find the distance $d^2 = 270^2 + 130^2 - 2 \times 270 \times 130 \times \cos 142^o$, $d = 381\,\text{km}$.

Let the angle opposite 130 km be $y°$, using the sine rule: $\dfrac{d}{\sin 142^o} = \dfrac{130}{\sin y}$, $y = 12^o$, so the bearing is $12 + 165 = 177^o$.

5 Draw the following diagram with the angles:



Hence, use the sine rule: $\sin x = \sin 27^o \times \frac{1750}{1000}$, $x = 53^o$, but this does not give the shortest distance. Hence, x = 180 – 53 = 127°. $B\hat{P}A = 180 - 27 - 127 = 26°$. Next, $AB = \frac{1000}{\sin 27^o} \times \sin 26 = 952\,\text{m}$. Bearing is 360 – (127 – 90) = 323°.

Exercise 1F

1 Use the sum of the angles for a triangle and the sine rule to determine the missing lengths and angles:

a 106 mm, $41.0^o, 87.0^o$

b $17.2\,\text{cm}, 17.2\,\text{cm}, 66^o$

c $11.1\,\text{cm}, 39.2^o, 87.8^o$

d
$$180 = 4x + 3x + 2x = 9x$$
$$x = 20°$$

Angles are: 40°, 60° and 80°

$16.3\,\text{m}, 22.0\,\text{m}$

2 Velina's method is correct; we cannot use the cosine rule in this case because the angle given is not between the two sides given. Solve $\sin\hat{C} = 40 \times \frac{\sin 35^o}{25}$, $\hat{C} = 66.6^o$ or $180 - 66.6 = 113.4°$

For $\hat{C} = 66.6^o$, $\hat{B} = 78.4^o$, $AC = \frac{25}{\sin 35^o} \times \sin 78.4^o = 42.7\,\text{cm}$. (3 s.f.)

For $\hat{C} = 113^o, \hat{B} = 31.6^o, AC = \frac{25}{\sin 35^o} \times \sin 31.6^o = 22.8\,\text{cm}$. (3 s.f.)

3 Use the cosine rule once to find one angle, then either the cosine rule or the sine rule to find a second angle. The third angle can be found by subtracting the other two from 180°:
$\hat{A} = 97.1^o, \hat{B} = 50.9^o, \hat{C} = 32.0^o$. (3 s.f.)

4 **a** Use the cosine rule:

$$\cos B = \frac{25 + 16 - 36}{2 \times 5 \times 4}$$
$B = 82.8^o$.

b
$$\frac{\sin A\hat{C}B}{5} = \frac{\sin 82.8°}{10}$$

$A\hat{C}B = 29.7°$ and $B\hat{A}C = 180 - 82.8 - 29.7 = 67.5°$

Then, $BC = \frac{10}{\sin 82.8^o} \times \sin 67.5^o = 9.3\,\text{cm}, DC = 5.3\,\text{cm}$.

Area $= \frac{1}{2} \times 10 \times 5.3 \times \sin 29.7° = 13.1\ \text{cm}^2$

5 Area $= \frac{1}{2} ac \sin B$

$$AB = c = \frac{2 \times \text{Area}}{a \sin B} = 20.0 \text{ cm}^2$$

6 Area of triangle ABC: $\frac{1}{2} AC \times BC \times \sin 20^o = 23.9 \text{cm}^2$

To find the area of triangle BCD, first find one of the angles using the cosine rule: $C\hat{B}D = 58.8°$, then area of triangle BCD: $\frac{1}{2} BD \times BC \times \sin C\hat{B}D = 47.9 \text{ cm}^2$.

Hence, the total area is 71.8cm^2.

Exercise 1G

1 Use the formulas for the length and area of the arc to find the following:

a **i** $l = 6.11 \text{cm}$ **ii** $A = 15.27 \text{cm}^2$

b **i** $l = 3.14 \text{cm}$ **ii** $A = 6.28 \text{ cm}^2$

c **i** $l = 23.82 \text{cm}$ **ii** $A = 125.07 \text{ cm}^2$

2 The angle between markings 12 and 5 is $\frac{5}{12} \times 360 = 150°$. The radius is 12.5 cm. Hence,

$l = \frac{150}{360} \times 2\pi \times 12.5 = 32.7 \text{cm}$

3 **a** Distance $l = \frac{200}{360} \times 2\pi \times 60 = 209 \text{m}$.

b Time in minutes $t = \frac{209.4}{0.26 \times 60} = 13.4 \text{min}$

c Time to make a full revolution $T = \frac{\pi d}{0.26 \times 60} = \frac{\pi \times 120}{0.26 \times 60} = 24.2 \text{min}$

4 **a** Angle $x = \frac{\text{Area} \times 360}{\pi r^2} = \frac{48\pi \times 360}{\pi \times 100} = 173°$

b Radius $x = \sqrt{\frac{8\pi}{\frac{50}{360}\pi}} = \sqrt{57.6} = 7.59 \text{cm}$ (3 s.f.)

5 Area of the shaded region is $A = \frac{\pi}{4} r^2 - \frac{1}{2} r^2$, hence

a $A = 18.3 \text{cm}^2$ **b** $A = 41.1 \text{cm}^2$. (3 s.f.)

6 Area of the shaded region is $A = \frac{55}{360} \pi r^2 - \frac{1}{2} r^2 \sin 55^o = 1.13 \text{ cm}^2$, (3 s.f.)

7 Area of the single triangle is $\frac{1}{2} \times 10.5^2 \times \sin 60^o$, hence total area covered by grass is:

$10.5^2 \pi - 3 \times 10.5^2 \times \sin 60^o = 59.9 \text{m}^2$.

Exercise 1H

1 a Volume of a cone $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi r^2 \times \pi r = 35000\,\text{cm}^3$ (3 s.f.)

b Hexagonal base is made up of 6 equilateral triangles each of side 5cm

Area of the hexagonal base $A = 6 \times \frac{1}{2} \times 5 \times 5 \times \sin 60^\circ = 64.95...$, volume of a regular prism

$V = Ah = 1490\,\text{cm}^3$ (3 s.f.)

c From the area of the circle, find the radius of the sphere $r = \sqrt{\frac{A}{\pi}} = 11.451797...\text{cm}$. Hence, the volume of the hemisphere $V = \frac{2}{3}\pi r^3 = 3150\,\text{cm}^3$ (3 s.f.)

2 a Find the volume of the container. Start by finding the area of an isosceles triangle: height of the triangle from Pythagoras theorem is $h = \sqrt{5.8^2 - 4.05^2} = 4.151806...\,\text{m}$. The area of the triangle $A = \frac{1}{2} 4.151806... \times 8.1 = 16.814817...\,\text{m}^2$, the volume of the container is $V = A \times 7.2 = 121\,\text{m}^3$.

b Height will decrease by $\Delta h = \frac{V}{5.83^2 \pi} = 1.13\,\text{m}$.

3 a i $V = \frac{4}{3}\pi r^3 = 1.41 \times 10^{-29}\,\text{m}^3$ (3 s.f.)

ii Radius of the earth $r = \frac{C}{2\pi}$, volume of the earth $V = \frac{4}{3}\pi\left(\frac{C}{2\pi}\right)^3 = \frac{1}{6\pi^2}C^3 = 1.09 \times 10^{21}\,\text{m}^3$ (3 s.f.)

iii $V = \frac{1}{6}\pi d^3$ where d is the diameter of the star, hence $V = 8.66 \times 10^{35}\,\text{m}^3$ (3 s.f.)

b The Earth compared to an atom.

$$\frac{V_{\text{Earth}}}{V_{\text{atom}}} = \frac{1.09 \times 10^{21}}{1.41 \times 10^{-29}} = 7.73 \times 10^{49}$$

$$\frac{V_{\text{UY Scuti}}}{V_{\text{Earth}}} = \frac{8.66 \times 10^{35}}{1.09 \times 10^{21}} = 7.95 \times 10^{14}$$

4 a Area of square base is $220^2 = 48400$.

Volume of pyramid $= \frac{1}{3}Ah = \frac{48400 \times 105}{3} = 1694000$

Volume of the material used $V_m = 0.96 \times 1694000 = 1626240 \approx 1630000\ \text{m}^3$

b Total weight = number of blocks $\times$ mass of a block

$\frac{V_m}{1.30 \times 1.30 \times 0.30} \times 2250 = 7.22 \times 10^9\,\text{kg}$.

5 Since the volume of a cylinder has square dependence on its radius, doubling the radius will result in 4 times larger volume, i.e. 400l. If the width is tripled, the volume becomes 900l.

Exercise 1I

1 a Area of the base $A_1 = \pi r^2$, lateral area $A_2 = 2\pi r\ h$, total area $A = 2A_1 + A_2 = 154\,\text{cm}^2$.

b Area of the base $A_1 = \pi r^2$, lateral area of the cone is $A_2 = \pi r\sqrt{r^2 + h^2}$, total area $A = A_1 + A_2 = 176\,\text{cm}^2$.

c Area of one triangle $A_1 = \dfrac{\sqrt{3}}{4}a^2$, surface area of the pyramid $A = 4A_1 = 501\,\text{mm}^2$

d A regular pentagon contains 5 equal triangles with angles of 72^o, $54^o, 54^o$. Use the sine rule to find the side length of an isosceles triangle to be $l = 2.042\,\text{cm}$, height of the triangle $h = 1.652$ cm, so that the area of the pentagon $A_1 = 9.91\,\text{cm}^2$. Next, find the slanting height of the pyramid: $h_2 = \sqrt{4.2^2 + 1.652^2} = 4.513\,\text{cm}$ so that the lateral area of the pyramid is $A_2 = 4.513 \times 5 \times 2.4 \times \dfrac{1}{2} = 27.08\,\text{cm}^2$. Total surface area is thus $A = A_1 + A_2 = 37.0\,\text{cm}^2$ (3 s.f.)

2 i $r = \sqrt{\dfrac{A}{4\pi}} = 6.46\,\text{cm}$

ii $h = \sqrt{\dfrac{2A}{3\pi}} = 10.6\,\text{cm}$

iii $A = \pi r^2 + \pi r\sqrt{r^2 + h^2}$, $r = \dfrac{C}{2\pi}$, $h = \sqrt{\left(\dfrac{A}{\pi} - r^2\right)^2 \times \dfrac{1}{r^2} - r^2} = 37.8\,\text{cm}$.

3 a i ratio $= \dfrac{A}{V} = \dfrac{4\pi r^2}{\frac{4}{3}\pi r^3} = \dfrac{3}{r}$.

ii $\dfrac{3}{r} > 955000$, $r < \dfrac{3}{955000}$, so the largest possible radius is $r = 3.14 \times 10^{-6}\,\text{m}$.

b i Cube with a side of length a has volume $a^3 = \dfrac{4}{3}\pi r^3$ and surface area $6a^2$. Hence, the ratio $= \dfrac{6}{a} = \dfrac{6}{\left(\frac{4}{3}\pi\right)^{\frac{1}{3}} r}$.

ii The cylinder with radius a and height $5a$ has volume $\dfrac{4}{3}\pi a^3 + 5\pi a^3 = \dfrac{19}{3}\pi a^3 = \dfrac{4}{3}\pi r^3$. The surface area of this cylinder is then $4\pi a^2 + 10\pi a^2 = 14\pi a^2$. Hence, the ratio = $\dfrac{42}{19a} = \dfrac{42}{19} \times \dfrac{1}{\left(\frac{4}{19}\pi\right)^{\frac{1}{3}} r}$.

iii Cube: $\dfrac{6}{\left(\frac{4}{3}\pi\right)^{\frac{1}{3}} r} \times \dfrac{r}{3} = \dfrac{2}{\left(\frac{4}{3}\pi\right)^{\frac{1}{3}}} = 1.24$, cylinder: $\dfrac{42}{19} \times \dfrac{1}{\left(\frac{4}{19}\pi\right)^{\frac{1}{3}} r} \times \dfrac{r}{3} = \dfrac{14}{19} \times \dfrac{1}{\left(\frac{4}{19}\pi\right)^{\frac{1}{3}}} = 0.85$.

4 **a** Area of one face = $\frac{1}{2}(1.2)^2\sin(60) = 0.6235\ldots\text{ cm}^2$

In a year, the total surface area painted on the dice will be:
$A = 0.6235\ldots \times 4 \times 3.5 \times 10^6 = 8.73 \times 10^6\text{ cm}^2$

b Volume of paint needed: $V_p = A \times 0.00254\text{ cm}^3$

Volume of one can: $V_c = \pi r^2 h = \pi(3.25)^2 \times 25.4\text{ cm}^3 \approx 842\text{cm}^3$

Number of cans is thus: $n = \frac{V_p}{V_c} = 26.3$, so 27 cans will be needed.

c The cost will be $16 \times 27 = \$432$.

5 Height of triangle DEF: $\sqrt{6^2 - 2^2} = 4\sqrt{2}$. Height of the triangle ACD:
$\sqrt{8^2 + \left(4\sqrt{2}\right)^2} = 4\sqrt{6} = 9.797\ldots$ Hence, the area of triangle ADC is $A = \frac{1}{2} \times 9.797\ldots \times 4 \approx 19.60\text{cm}^2$.

6 **a** $V = 6 \times 11 \times 15 + \frac{1}{2} \times 5 \times 11 \times 15 = 1400\text{ m}^3$,

$A = 2 \times \left(6 \times 11 + 6 \times 15 + \frac{1}{2} \times 5 \times 11 + \sqrt{5^2 + 5.5^2} \times 15\right) + 11 \times 15 = 755\text{ m}^2$, both correct to 3 s.f.

b $V = \pi \times 3^2 \times 9 + \frac{1}{3} \times \pi \times 3^2 \times \sqrt{4^2 - 3^2} = 279\text{ cm}^3$, $A = 2\pi \times 3 \times 9 + \pi \times 3^2 + \pi \times 3 \times 4 = 236\text{ cm}^2$, both correct to 3 s.f.

7 **a** $V = \frac{2}{3}\pi 3^3 + \pi \times 3^2 \times 12 = 396\text{ m}^3$.

b Surface area of the silo $A = 2\pi \times 3 \times 12 + \frac{1}{2} \times 4\pi \times 3^2 = 283\text{m}^2$, so $\frac{383}{8.5} = 33.3$ l of paint will be needed.

8 Volume of the full cube is 6^3 cm^3.

Volume of the pyramid removed is $\frac{1}{3} \times A \times h = \frac{1}{3} \times \left(\frac{1}{2} \times 3 \times 3\right) \times 3 = 4.5\text{cm}^3$, so the new volume is $6^3 - 4.5 = 211.5\text{cm}^3$.

Area of the full cube is $6 \times 6^2\text{ cm}^2$

Surface area of each face removed from the corner, is $\frac{1}{2} \times 3^2$ as they are right-angled isosceles triangles.

Triangle IJK is equilateral with side length d where $d^2 = 3^2 + 3^2 = 18$

Surface area of triangle IJK is $\frac{1}{2} \times \sqrt{18} \times \sqrt{18} \times \sin 60 \approx 7.79$

So new area is $6^3 - 3\times\frac{1}{2}\times 3^2 + 7.79... \approx 210\,\text{cm}^2$.

Chapter review

1 **a** **i** 0.2 **ii** 0.154

b **i** 2.3 **ii** 2.30

c **i** 2.0 **ii** 1.99

d **i** 1.8 **ii** 1.83

e **i** 0.2 **ii** 0.248

2 **a** Percentage error = $\left|\frac{15896255521782636000 - 1.5\times 10^{19}}{15896255521782636000}\right|\times 100\% = 5.64\%$.

b 13.82 billion years = 4.358×10^{17} s (not including leap days!), so there is not enough time to write down all the possible settings.

3 Use your GDC to plot $y = x^3 + 3x$ and $y = 5$. Find their intersection point to be at $x = 1.15417$. Use given formula to find the intersection with more precision: take $p = 3$, $q = 5$ to obtain x = 1.154171495. The percentage error of graphical method is $1.3\times 10^{-4}\,\%$.

4 **a** Cosine rule gives: $QR^2 = PR^2 + PQ^2 - 2PQ\times PR\times\cos 31^o$, thus solving gives $PR = 24.8$ cm.

b First, find the angle R: $\sin R = \frac{\sin 31^o}{15}\times 13.4$, $R = 27.3936...^o$, so $Q = 121.6063...^o$. Apply the sine rule again to find $PR = \frac{15}{\sin 31^o}\sin Q = 24.8\,\text{cm}$.

5 Difference between the heights of the Rockefeller Centre and the Empire State Building is 61 m, so $\tan\beta = \frac{61}{1310}, \beta = 2.67^o$.

Difference between the heights of One World Trade Center and the Empire State is 62.2 m, so $\tan\beta = \frac{62.2}{4600}, \beta = 0.775^o$.

6 **a** The range of possible values of the lengths of the sides is $229.5\,\text{m} \le l < 230.5\,\text{m}$, so the range of possible values of the area of the base is $52670.25\ \text{m}^2 \le A < 53130.25\ \text{m}^2$.

b The range of possible values of the area of the football pitch is $6400\,\text{m}^2 \le A < 8250\,\text{m}^2$, then if we only consider the ratio of the area of the football pitch to the area of the base, the range of possible values is: $\frac{52670.25}{8250} \le R < \frac{53130.25}{6400}, 6.38 \le R < 8.30$. If we consider complete pitches, $P = 2\times 3 = 6$.

7 Let the length of the hour hand be x

$$s = \frac{d}{t} \Rightarrow d = st$$
$$t = 12 \text{ hours} = 43200 \text{ seconds}$$
$$s = 3\times 10^8 \text{ m/s}$$
$$x = \frac{d}{2\pi} = \frac{43200\times 3\times 10^8}{2\pi} = 2.063\times 10^{12} \text{ m} = 2.063\times 10^9 \text{ km}$$

8 **a** The quick way to do this question is to realise that because the formula for a cone is one third of the formula for the cylinder you must remove $\frac{2}{3}$ of the cylinder so $\alpha = 240^\circ$

This can be verified algebraically

Volume of the cone $V_1 = \frac{1}{3}\pi r^2 h$, volume of the modified cylinder $V_2 = \pi r^2 h - \frac{\alpha\pi}{360} r^2 h$. For volumes to be equal, $\pi r^2 h - \frac{\alpha\pi}{360} r^2 h = \frac{1}{3}\pi r^2 h \Rightarrow \alpha = 240^\circ$

b Lateral area of the modified cylinder is $\left(\frac{120}{360}\times 2\times\pi + 2\right) rh = \left(\frac{2}{3}\pi + 2\right) rh$. Lateral area of the cone is $\frac{\pi rh}{\sin\theta}$ as $l = \frac{h}{\sin\theta}$ For them to be equal, $\sin\theta = \frac{\pi}{\left(\frac{2}{3}\pi + 2\right)}, \theta = 50^\circ$.

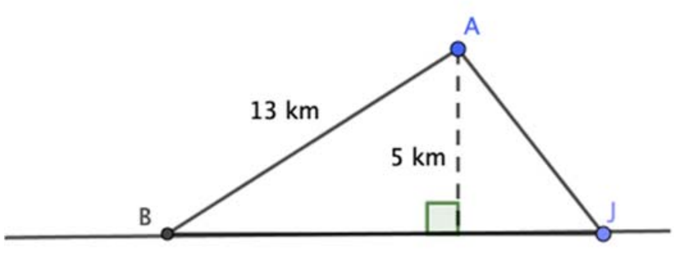
9 **a** $BM = \frac{1}{2}EF = 115.17828\,\text{m},\ VM = 186.47285\,\text{m},\ \hat{M} = 51.853975^\circ, \hat{V} = 38.146025.$

b $BF = \sqrt{2}BM = 162.88668\,\text{m},\ VF = 219.17609\,\text{m},\ \hat{F} = 41.99723^\circ,\ \hat{V} = 48.00277^\circ.$

c $VE = VF = 219.17609\,\text{m},\ \hat{E} = \hat{F} = 58.29771^\circ,\ \hat{V} = 63.40458^\circ,\ A = 21477.621\,\text{m}^2.$

d $VB^2 = 21506.088\,\text{m}^2$, so the percentage error $= \left|\frac{VB^2 - A}{A}\right| \times 100\% = 0.133\%.$

10 a Suppose the distance between Betatown and J is d and let M be where the perpendicular from A meets BJ.



|MB| = 12 km by Pythagoras, so |MJ| = d – 12. $\left|AJ\right| = \sqrt{5^2 + (d-12)^2}$.

Time taken:

$$t = \frac{\sqrt{5^2 + (d-12)^2}}{70} + \frac{d}{110}.$$

b Suppose the angle is θ. $|AJ| = \frac{5}{\sin\theta}$ and $|MJ| = \frac{5}{\tan\theta}$. Thus, $|BJ| = 12 + \frac{5}{\tan\theta}$.

Then, time taken is $t = \frac{5}{\sin\theta} \times \frac{1}{70} + \left(12 + \frac{5}{\tan\theta}\right) \times \frac{1}{110}$

Exam-style questions

11 a $\frac{5.5}{0.25}$ ohms $\le R < \frac{6.5}{0.15}$ ohms, 22 ohms $\le R <$ 43 ohms.

b Maximum percentage error $= \frac{43-30}{30} \times 100\% = 43\%$.

12 a $12x^4$ **b** x^2 **c** $\frac{9y^4}{49x^6}$.

13 a 1.12×10^{-27} **b** 1 : 1840.

c percentage error $= \left|\frac{10^{-30} - 9.109 \times 10^{-31}}{9.109 \times 10^{-31}}\right| \times 100\% = 9.78\%$.

14 Number of bearings $= \frac{\pi \times 12^2 \times 50}{\frac{4}{3}\pi \times 2^3} = 675$.

15 a $\frac{A\hat{O}B}{360} \times 2 \times \pi \times 15 = 10 \Rightarrow A\hat{O}B = \frac{120}{\pi}$, $B\hat{O}C = 180 - \frac{120}{\pi} = 141.8°$.

Shaded area $= \frac{B\hat{O}C}{360} \times \pi \times BO^2 - \frac{1}{2}BO^2 \sin B\hat{O}C = 209\,\text{cm}^2$.

b Length of arc $= \frac{141.8}{360} \times 2 \times \pi \times 15 = 37.12388\ldots$

From cosine rule, length of chord $BC = \sqrt{15^2 + 15^2 - 2 \times 15 \times 15 \times \cos(141.8)} = 28.34870\ldots$

Perimeter of the shaded region = length of arc + length of chord = 65.5 cm

16 $A = \frac{1}{2}x(x+1)\sin 60^o = 14\sqrt{3}$

Solve on a GDC to get $x = 7$. Take the positive solution. Use the cosine rule to find $BC = \sqrt{7^2 + 8^2 - 2 \times 7 \times 8 \times \cos 60^o} \approx 7.55$ so the perimeter is $15 + 7.55 \approx 22.5\,\text{cm}$.

17 a Let the centre of the face ABCD be O. Then,

$AO = \frac{1}{2}\sqrt{16^2 + 16^2} = 11.31\ldots\text{cm}$, so $\cos\alpha = \frac{11.31\ldots}{20}$, $\alpha = 55.55 \approx 56^o$.

b Slanting height of the pyramid is $\sqrt{20^2 - 8^2} = \sqrt{336}$, so $\cos\beta = \frac{8}{\sqrt{336}}$, $\beta = 64^o$.

c $A\hat{E}O = 90^o - E\hat{A}O = 34.45^o$, $so\ A\hat{E}C = 68.9^o$.

d $EO = \sqrt{20^2 - 11.31..^2} = \sqrt{400 - 128} = \sqrt{272} = 4\sqrt{17}$ cm,

$V = \frac{1}{3} \times 16^2 \times 4\sqrt{17} = 1407\,\text{cm}^3$.

e $A = 16^2 + 4 \times \frac{1}{2} \times 16 \times \sqrt{336} = 843\,\text{cm}^2$.

2 Representing and describing data: descriptive statistics

Skills Check

1 Mean $= \frac{14+15+17+22+26+22+21+16+17+22}{10} = \frac{192}{10} = 19.2$

Median $= \left(\frac{n+1}{2}\right)^{th} = (5.5)^{th} = \frac{17+21}{2} = 19$

Mode is the commonest number in the list: 22

Exercise 2A

1 Use quota sampling; take at random n students' marks from each of the five year groups

2

a

x	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$f(x)$	10	8	16	13	21	9	10	9	5	4	4	3	3	2	1	2

b $\frac{120}{5} = 24$ dogs were selected

c

x	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Sample size	4	3	7	5	9	4	4	4	2	2	2	1	1	1	0	1

3 Stratified sampling is important to get an accurate representation of the population. As different sectors of society may vote differently to each other, it is important to have those sectors represented proportionately. Important strata may include age, socioeconomic class, income, gender, education level or geographical region.

Exercise 2B

Note: Answers can also be obtained directly from the GDC

1 a Mean $\left(= \frac{1.0+1.5+2.3+\cdots+7.5+17.8+25.0}{25} = \frac{151.7}{25}\right) = 6.068$

Median $\left(= \left(\frac{n+1}{2}\right)^{th} = (13)^{th}\right) = 5.2$

Mode $= 7.5$

17.8 and 25.0 may be outliers

b Mean $\left(=\frac{1.1+2.2+2.5+\cdots+4.6+4.9+6.1}{15}=\frac{53}{15}\right)=3.533$

Median $\left(=\left(\frac{n+1}{2}\right)^{th}=(8)^{th}\right)=3.6$

Mode $=2.5$

c Mean $\left(=\frac{22+39+45+\cdots+91+95+98}{20}=\frac{1293}{20}\right)=64.65$

Median $\left(=\left(\frac{20+1}{2}\right)^{th}=(10.5)^{th}=\frac{62+62}{2}\right)=62$

Mode $=62$

2 **a** Mean $\left(=\frac{0\times5+1\times5+2\times6+3\times2+5\times1+8\times1}{5+5+6+2+1+1}=\frac{36}{20}\right)=1.8$

Median $\left(=\left(\frac{20+1}{2}\right)^{th}=(10.5)^{th}=\frac{1+2}{2}\right)=1.5$

Mode $=2$

b Standard deviation $=1.89$

Variance = (standard deviation)^2 $=3.56$

3 **a** Mean $\left(=\frac{0\times3+4\times2+6\times8+8\times4+10\times2+12\times12+14\times5}{3+2+8+4+2+12+5}=\frac{322}{36}\right)=8.9444$

Median $=10$

Mode $=12$

The most appropriate measure to use would be the mode

b Standard deviation $=4.0957$

Hence, the average distance from the mean is approximately 4

c Range $=14-0=14$

From the GDC $IQR = Q_3 - Q_1 = 12 - 6 = 6$

The data is mainly concentrated within the middle range as the IQR is a lot smaller than the range

4 **a** Mean basketball $= 200.8$

Mean men $= 172.27$

Standard deviation basketball $= 10.4575$

Standard deviation men $= 10.3116$

b On average basketball players are taller by approx. 28cm; however, they vary in height by the same amount

5 Robotics:

Mean $= 76.4545$

Median $\left(= \left(\frac{n+1}{2}\right)^{th} = (6)^{th} \right) = 81$

Standard deviation $= 12.9221$

$Q_1 = 65$

$Q_3 = 84$

Astronomy:

Mean $= 69.4286$

Median $\left(= \left(\frac{n+1}{2}\right)^{th} = (6)^{th} \right) = 69$

Standard deviation $= 7.8803$

$Q_1 = 61$

$Q_3 = 77$

The calculations partly support Mr Jones' claim as the mean and median are both higher for the robotics club, but the standard deviation is much lower for Astronomy students, and in fact the two lowest scoring students both do Robotics.

6 Answers will vary

7 Mean $= 2 \times 30 = 60$
Standard deviation $= 2 \times 3 = 6$

8 **a** Mrs Ginger:

Mean $= 2 \times 32 + 20 = 84$

Standard deviation $= 2 \times 8 = 16$

Mr Ginger:

Mean $= 2.5 \times 32 = 80$

Standard deviation $= 2.5 \times 8 = 20$

Miss Ginger:

Mean $= 3 \times 32 - 20 = 76$

Standard deviation $= 3 \times 8 = 24$

b Matty:
Mrs Ginger: $12 \times 2 + 20 = 44$
Mr Ginger: $12 \times 2.5 = 30$
Miss Ginger: $12 \times 3 - 20 = 16$
Zoe:
Mrs Ginger: $25 \times 2 + 20 = 70$
Mr Ginger: $25 \times 2.5 = 62.5$
Miss Ginger: $25 \times 3 - 20 = 55$
Ans:
Mrs Ginger: $36 \times 2 + 20 = 92$
Mr Ginger: $36 \times 2.5 = 90$
Miss Ginger: $36 \times 3 - 20 = 88$

c For all students, Mrs Ginger's methodology gives them the highest mark. For students with high marks, the methods give relatively similar marks. For middling students, the methods vary a bit. For low scoring students, the methods vary wildly, indeed, if a student had a mark of 6 or lower, their new mark would be negative with Miss Ginger's methodology.

Exercise 2C

1 **a** **i** Modal class is $150 \le n < 180$

ii Mean $= 111.625$

iii Median $= 105$

b **i** Modal class is $50 \le s < 55$

ii Mean $= 54.417$

iii Median $= 52.5$

c **i** Modal class is $7 \le t < 8$

ii Mean $= 5.864$

iii Median $= 5.5$

The GDC treats the data as discrete taking each value as being equal to the mid-point. This means the median will always be in the middle of the class in which it lies. In part **a** for example the median value is the value of the 120$^{\text{th}}$ car. This is likely to be near the high end of the class 90 to 120 which contains the 80th to the 121st car, so the middle of the class is unlikely to be a good estimate.

2 **a** The modal class is $30 \le c < 40$

b Mean $= 34$

c Standard deviation $= 10.116$
The average amount of money spent more or less than the average amount was £10.12

d Variance $= 10.116^2 = 102.333$
Range $= 60 - 10 = 50$

IQR $= 35 - 25 = 10$
These are all estimates as they assume all the data is at the midpoints of the classes.

3 **a** The modal class is $180 \le x < 190$

b Mean $= 180.2$
Standard deviation $= 10.998$

The average distance from the average height is approx. 11cm

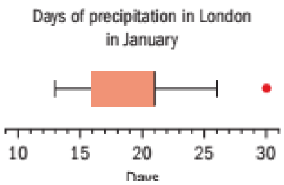
4 **a** Male mean $= \$2546$
Female mean $= \$2115$

Male standard deviation $= \$730$
Female standard deviation $= \$635$

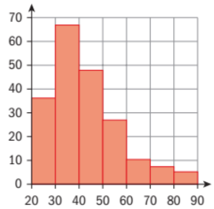
b On average, men earned more than women and male income varied more than female income.

Exercise 2D

1 $Q_1 = 16$, $Q_3 = 21$, $IQR = 21 - 16 = 5$, so outliers are any points below $16 - 1.5 \times 5 = 8.5$ or above $21 + 1.5 \times 5 = 28.5$.



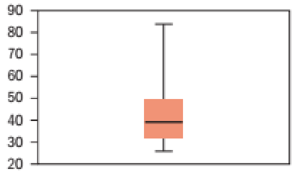
2 **a**



b Mean $= 42.5$

c $IQR = 49.8 - 32.1 = 17.7$, so outliers are points below $32.1 - 1.5 \times 17.7 = 5.6$ or above $49.8 + 1.5 \times 17.7 = 76.4$ indicating that there are at least 5 outliers, and an estimated 7 outliers

These cannot be shown on a box and whisker plot as their values are not known.



3 **a**

b The data heavily skewed to the right (or is negatively skewed), so non-symmetric

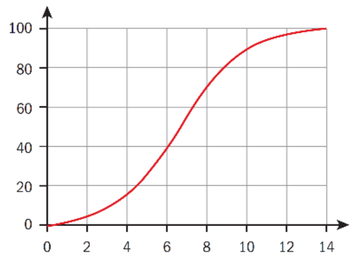
4 Student's own answers

Exercise 2E

1 **a**

Time, x minutes	Frequency	Cumulative Frequency
$0 \le x < 2$	5	5
$2 \le x < 4$	11	16
$4 \le x < 6$	23	39
$6 \le x < 8$	31	70
$8 \le x < 10$	19	89
$10 \le x < 12$	8	97
$12 \le x < 14$	3	100

b



c From the graph, $Q_1 = 4.9$, median $= Q_2 = 6.7$, $Q_3 = 8.4$, so $IQR = 8.4 - 4.9 = 3.5$

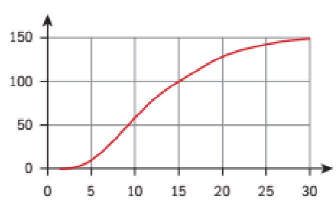
d From the graph, the 10th percentile is 3.1

e From the graph, 94 people waited less than 11 minutes, so 6 can claim for a refund for their fare

2 a

Number of words, x	Frequency	Cumulative Frequency
$0 \le x < 4$	5	5
$4 \le x < 8$	32	37
$8 \le x < 12$	41	78
$12 \le x < 16$	28	106
$16 \le x < 20$	22	128
$20 \le x < 24$	12	140
$24 \le x < 28$	7	147
$28 \le x < 32$	3	150

b

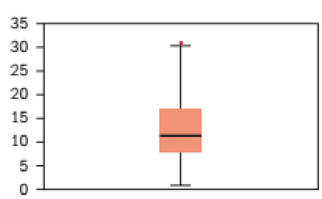


c From the graph, $Q_1 = 8$, median $= Q_2 = 11.5$, $Q_3 = 17$, so $IQR = 17 - 8 = 9$

d Points are outliers if their value is less than $8 - 1.5 \times 9 = -5.5$ or more than $17 + 1.5 \times 9 = 30.5$ indicating that there may be up to 3 outliers in the group $28 \le x < 32$.

e From the graph, the 90th percentile is 22

f (assuming only one outlier):

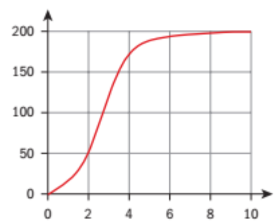


g A book for adults because it is more linguistically advanced than a children's book.

3 a

Height, h (m)	Frequency	Cumulative Frequency
$0 \le h < 1$	17	17
$1 \le h < 2$	35	52
$2 \le h < 3$	69	121
$3 \le h < 4$	51	172
$4 \le h < 6$	22	194
$6 \le h < 10$	6	200

b



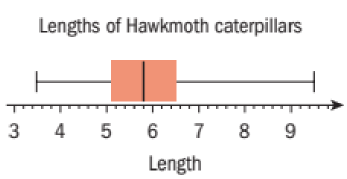
c From the graph, the median is 2.7m

d From the graph, $Q_1 = 2$ and $Q_3 = 3.5$, so $IQR = 3.5 - 2 = 1.5$

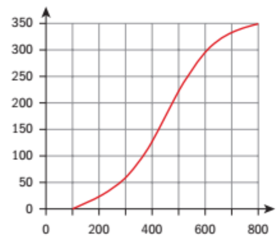
e The 10th percentile is 1.1m

f The smallest of the 12 trees would be 4.9m tall

4



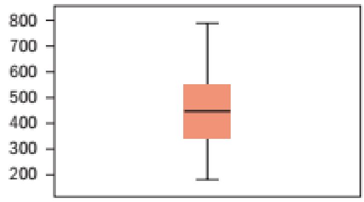
5 a



b From the graph, $Q_1 = 345$, median $= Q_2 = 450$, $Q_3 = 550$, so $IQR = 550 - 345 = 205$

c Points are outliers if their value is less than $345 - 1.5 \times 205 = 37.5$ or more than $550 + 1.5 \times 205 = 857.5$ indicating that there are no outliers

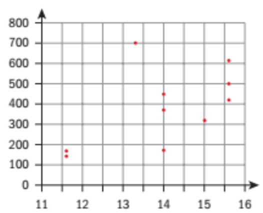
d



e The attraction loses revenue on 90 days per year

Exercise 2F

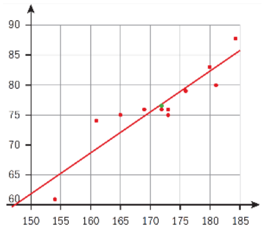
1 a



b There is a weak positive correlation

c There is a slight increase in price associated with an increase in screen size

2 a



b $(\bar{h}, \bar{w}) = (171.636, 76.636)$ shown on the plot above in green

c Drawn on the plot above

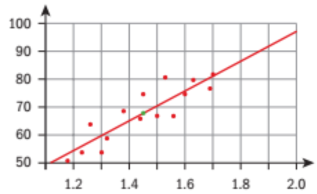
d 75kg

e From the GDC we obtain $r = 0.9150$

f There is a strong positive correlation, indicating that the taller the player, the heavier they are on average

g This is likely a causation because if somebody is taller, there is more body mass and thus they would likely weigh more

3 a



b $\left(\bar{h},\bar{v}\right)=\left(1.451, 68.067\right)$ shown on the plot above in green

c Drawn on the plot above

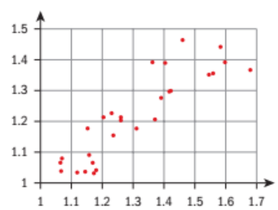
d From the graph, one would expect Klaus to score 93%. However, it is not appropriate to use this extrapolated result as 1.9 is far outside the current data range

e $r=0.884$

f There is a strong positive correlation

g This indicates that people who are taller scored better on the vocabulary test. From the heights, it is likely that the people surveyed were children. Older children would likely be taller and would be expected to have a better vocabulary.

4 a



b $r=0.8873$

There is a strong positive correlation between the price of unleaded and the price of diesel.

Chapter review

1 a Male mean $=13.317$

Male standard deviation $=3.374$

Female mean $=15.850$

Female standard deviation $=4.228$

On average, females are slower than males, however the time taken varies more for women than men

b Mean $\left(=\frac{60}{100}\times\frac{799}{60}+\frac{40}{100}\times\frac{634}{40}=\frac{1433}{100}\right)=14.330$

Standard deviation $=3.940$

c Answers will vary

d For example, take every 2.5th element of data, so take alternatively every second and third value. Answers for mean and standard deviation will vary depending on exact method used and first value chosen.

e Randomly select 24 male swimmers and 16 female swimmers. Answers for mean and standard deviation will vary.

f Answers will vary

2 **Note: the answers for the mean and median can be obtained directly from the GDC.**

a Mean $\left(=\frac{7+23+32+\cdots+56+45+32}{15}=\frac{822}{15}\right)=54.8$ cm

Median $\left(=\left(\frac{n+1}{2}\right)^{th}=(8)^{th}\right)=56\,\text{cm}$

Mode is the most common element of the list, 32 cm
The most appropriate measure is the median because the data are not symmetrical (positively skewed).

b Mean $\left(=\frac{46+54+58+\cdots+185+270+300}{12}=\frac{1467}{12}\right)=\$122.25$

Median $\left(=\left(\frac{n+1}{2}\right)^{th}=(6.5)^{th}=\frac{79+96}{2}\right)=\$87.5$

Mode is the most common element of the list, \$62
The most appropriate measure is the median because the data are not symmetrical (positively skewed).

c Mean $\left(=\frac{4+7+6+\cdots+8+6+7}{19}=\frac{122}{19}\right)=6.421$ hours

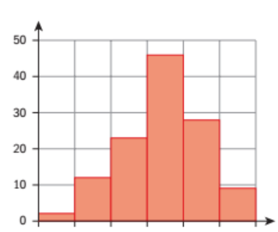
Median $\left(=\left(\frac{n+1}{2}\right)^{th}=(10)^{th}\right)=6\,\text{hours}$

Mode is the most common element of the list, 6 hours
The most appropriate measure is the mean because the data are approximately symmetrical.

3 **a** The modal class is $50 \le l < 60$

b Median $= 55$ cm
Mean $= 54.417$ cm
Standard deviation $= 11.277$ cm

c



4 **a** Mean $\left(=\frac{12000+13000+15000+17500+21000}{5}=\frac{78500}{5}\right)=\$15700$

b Standard deviation $=\$3250$ (3 sig figs)

c $\frac{13000-12000}{12000}\times100=8.33\%$ increase

d **i** The *y*-axis does not start at zero. Therefore, the graph is misleading because it makes it look like the profit was multiple times higher in each year.

ii It is drawn like this to make it look like the profits are increasing faster than they actually are.

5 **a** Mean $=32.8$

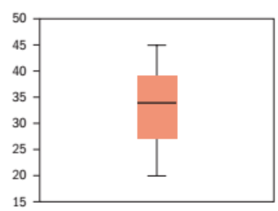
Standard deviation $=7.505$

b Range $=45-20=25$

$IQR=Q_3-Q_1=39-27=12$

c Median $\left(=\left(\frac{n+1}{2}\right)^{th}=(18)^{th}\right)=34$. Outliers are points that are smaller than $27-12\times1.5=9$ or bigger than $39+12\times1.5=57$, so no outliers.

d



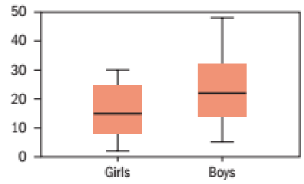
6 **a** Girls

Mean $=16.24$
Median $=15$
$Q_1=8$, $Q_3=24.5$, $IQR=24.5-8=16.5$, range $=30-2=28$. Girls are outliers if they do fewer than $8-1.5\times16.5=-16.75$ or more than $24.5+1.5\times16.5=49.25$ press-ups, therefore there are no outliers.

Boys

Mean $=24.48$
Median $=22$
$Q_1=13.5$, $Q_3=33$, $IQR=33-13.5=19.5$, range $=48-5=43$. Boys are outliers if they do fewer than $13.5-1.5\times19.5=-15.75$ or more than $33+1.5\times19.5=62.25$ press-ups, therefore there are no outliers.

b

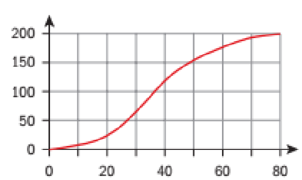


c Boys do on average more press-ups and have a larger range, with a longer whisker in the top half of the plot, indicating that the data is more skewed.

7 **a**

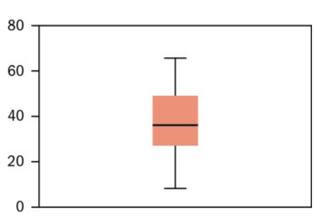
Number of hours, x	**Frequency**	**Cumulative Frequency**
$0 \le x < 10$	8	8
$10 \le x < 20$	16	24
$20 \le x < 30$	41	65
$30 \le x < 40$	54	119
$40 \le x < 50$	36	155
$50 \le x < 60$	22	177
$60 \le x < 70$	17	194
$70 \le x < 80$	6	200

b



c From the graph: Median = 36, $IQR = Q_3 - Q_1 = 49 - 27 = 22$

d



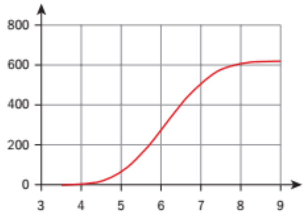
8 **a** Mean = 6.13 min

Standard deviation = 0.921 min

b

Time (min), x	**Frequency**	**Cumulative Frequency**
$3.5 \le x < 4$	6	6
$4 \le x < 4.5$	14	20

$4.5 \le x < 5$	48	68
$5 \le x < 5.5$	89	157
$5.5 \le x < 6$	121	278
$6 \le x < 6.5$	129	407
$6.5 \le x < 7$	103	510
$7 \le x < 7.5$	70	580
$7.5 \le x < 8$	30	610
$8 \le x < 8.5$	10	620
$8.5 \le x < 9$	2	622



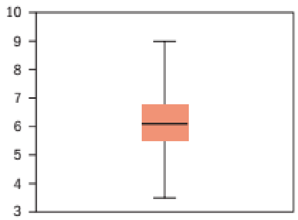
c **i** Median $= 6.1$

ii $Q_1 = 5.5$, $Q_3 = 6.75$

iii $IQR = Q_3 - Q_1 = 6.75 - 5.5 = 1.25$

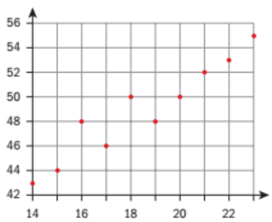
iv The 85th percentile is 7.1

d



e Points are outliers if they are below $5.5 - 1.5 \times 1.25 = 3.625$ or above $6.75 + 1.5 \times 1.25 = 8.625$ so there is a possibility that there are outliers at both ends of the data

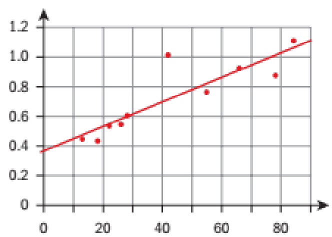
9 **a**



b There is a strong positive correlation

c Owing to the strong positive correlation, it is likely that by increasing the temperature, the number of eggs increases

10 a



b The point (42,1.02) is a likely outlier because of the distance from the line of best fit compared to the others

c The mean point is (43.333,0.6978)

d Line drawn on plot above

e $r = 0.9732$

f This is a very strong positive correlation, indicating that as one gets older, reaction times get slower

Exam style questions

11 a Range = 20 − 4 = 16

b Mean = 13.4

c Median = 13.5

d The modal mark (the most common mark) is 12

e Standard deviation $= 4.897...$ Variance $= (4.897...)^2 = 24.0$

12 a Mean = 23.6℃

b Standard deviation = 3.38℃

c Mean = 22.8℃

d Standard deviation = 5.52℃

e The mean temperature in Tenerife (23.6°C) is higher than that of Malta (22.8°C) while the temperature in Malta varies more than in Tenerife as the standard deviation is larger (5.52 and 3.38).

13 a Median = 69 min

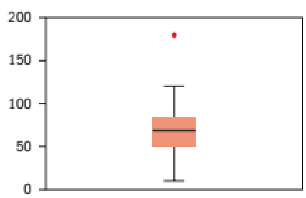
b $Q_1 = 50$ min

c $Q_3 = 83\,\text{min}$

d Range $= 180 - 10 = 170$ min

e $IQR = 83 - 50 = 33$ min. Outliers are points that are smaller than $50 - 1.5 \times 33 = 0.5$ or larger than $83 + 1.5 \times 33 = 170$ indicating that 180 is an outlier.

f



14 a

Height, (x cm)	Frequency
$20 \le x < 25$	3
$25 \le x < 30$	3
$30 \le x < 35$	4
$35 \le x < 40$	7
$40 \le x < 45$	4
$45 \le x < 50$	2
$50 \le x < 55$	1

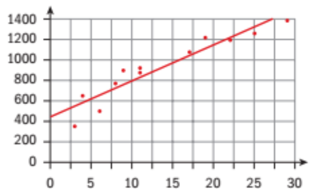
b Estimated mean $= 35.8$ cm

c Estimated standard deviation $= 7.99$ cm

d Estimated variance $= 63.9$ cm

e On average, the plants in Eve's neighbour's garden are 3.7 cm shorter than in Eve's garden whilst the plants in both gardens vary by a similar amount.

15 a



b Mean temp $= 13.7$°C

Mean sales $= \$927$

line of best fit drawn on plot above

c $r = 0.944$

The PMCC is close to one, indicating a strong positive correlation.

d When the temperature is higher, sales of ice cream are higher

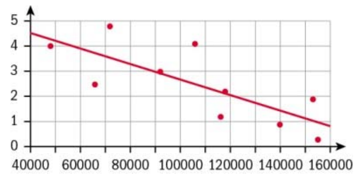
16 a Mean $= 9.6 \times 200 + 5000 = 6920\,\text{kg}$

b Standard deviation $= 2.15 \times 200 = 430\,\text{kg}$

17 a $r = -0.766$

This is a strong negative correlation.

b



Mean salary $= \$106600$

Mean distance $= 2.49\,\text{km}$

The mean point is $(\bar{s}, \bar{d}) = (106600, 2.49)$. The line of best fit should pass through this point.

c The extrapolation is for much further than the data provided and the decrease in salary with distance may not be linear.

3 Dividing up space: coordinate geometry, Voronoi diagrams, vectors, lines

Skills check

1 $d=\sqrt{(1-5)^2+(2-4)^2}=\sqrt{16+4}=\sqrt{20}=4.47$

2 $\left(\frac{1+5}{2},\frac{2+4}{2}\right)=(3,3)$

3 a $\cos(40)=\frac{x}{10}$. Hence, $x=7.66$.

$\frac{y}{10}=\sin(40)$. Hence, $y=6.43$.

b $\tan(x)=\frac{6}{9}$. Hence, $x=\arctan\left(\frac{6}{9}\right)=33.7^o$.

4 a i $\frac{x}{50}=\sin(60)$. Hence, $x=43.3\text{km}$ east of A.

ii $\frac{y}{50}=\cos(60)$. Hence, $y=25.0\text{km}$ north of A.

b $x=270-(180-90-60)=270-30=240^o$.

Exercise 3A

1 a i $d=\sqrt{((2-1)^2+(1--4)^2}=\sqrt{1+25}=\sqrt{26}=5.10$

ii $d=\sqrt{(2-0)^2+(4-3)^2+(-3+2)^2}=\sqrt{4+1+1}=\sqrt{6}=2.45$

b i $m=\left(\frac{2+1}{2},\frac{(1-4)}{2}\right)=(1.5,\ -1.5)$.

ii $m=\left(\frac{2+0}{2},\frac{(4+3)}{2},\frac{-3-2}{2}\right)=(1,3.5,-2.5)$.

2 a i $AB=\sqrt{(21-(-3))^2+((-13)-14)^2}=\sqrt{576+729}=\sqrt{1305}=36.1$

ii $AB=\sqrt{(-17-(-2))^2+(11-8)^2+(0-(-12))^2}=\sqrt{378}=19.4$

b i $m=\left(\frac{21-3}{2},\frac{14-13}{2}\right)=(9,0.5)$

ii $m=\left(\frac{-17-2}{2},\frac{11+8}{2},\frac{0-12}{2}\right)=(-9.5,9.5,-6)$.

3 a $d=\sqrt{(26-20)^2+(31-25)^2+(12-11)^2}=\sqrt{36+36+1}=\sqrt{73}=8.54\text{ km}$

b $d_1 = \sqrt{20^2 + 25^2 + 11^2} = \sqrt{1146} = 33.9 < 40.$ The first aircraft can be detected.

$d_2 = \sqrt{26^2 + 31^2 + 12^2} = \sqrt{1781} = 42.2 > 40.$ The second aircraft cannot be detected.

4 **a** $A = (250, 0, 0), B = (250, 400, 0), C = (0, 400, 60), D = (0, 0, 60).$

b $M = \left(\frac{250+0}{2}, \frac{400+400}{2}, \frac{0+60}{2}\right) = (125, 400, 30).$

c $l = \sqrt{(250-125)^2 + (0-400)^2 + (0-30)^2} + \sqrt{(125-0)^2 + (400-0)^2 + (30-60)^2}$

$= \sqrt{176525} + \sqrt{176525} = 2\sqrt{176525} \approx 840\,\text{m}$

5 **a** $AB = \sqrt{(340-97)^2 + (77-(-139))^2 + (21-21)^2} = \sqrt{59049 + 46656 + 0} = \sqrt{105705}.$

If the length of the base is x then $2x^2 = AB$

length of a side of the base$=\frac{\text{AB}}{\sqrt{2}}=\sqrt{\frac{105705}{2}}.$

Area of the base $= \left(\frac{AB}{\sqrt{2}}\right)^2 = \frac{105705}{2} = 52900\ \text{m}^2.$

b volume $= \frac{\text{area of base} \times \text{height}}{3} = \frac{52900 \times 138}{3} = 2\,430\,000\,\text{m}^3$

c midpoint of $AB = \left(\frac{340+97}{2}, \frac{77-139}{2}, \frac{-21-21}{2}\right) = (218.5, -31, -21).$

Vertex is 138m above this, so at $(218.5, -31, 117)$.

d $d = \sqrt{(340-218.5)^2 + (-139-(-31))^2 + (-21-117)^2} = 213\,\text{m}$

Exercise 3B

1 **a** $x = 2$ **b** $y = 6$ **c** $(2, 6)$

2 **a** $y = 3x + 5$ **b** $y = -2x + 0.4$ **c** $y = 4.5x + 5$

d $y = 2x + c.$

$5 = 2 \times 3 + c.$

$c = -1.$

$y = 2x - 1.$

3 **a** $m = \frac{-3-5}{5-3} = -4.$ $5 = 3 \times -4 + c.$ $c = 17.$ $y = -4x + 17.$

b $m = \frac{4--1}{3-2} = 5.$ $4 = 5 \times 3 + c.$ $c = -11.$ $y = 5x - 11.$

c $m = \frac{4-2}{3--3} = \frac{2}{6} = \frac{1}{3}.$ $4 = \frac{1}{3} \times 3 + c.$ $c = 3.$ $y = \frac{1}{3}x + 3.$

4 **a** $m = \frac{8--4}{6-12} = -\frac{12}{6} = -2.$ $8 = -2 \times 6 + c.$ $c = 20.$ $y = -2x + 20.$

b **i** $y = -2 \times 0 + 20 = 20.\ (0, 20).$

ii $y = -2 \times 22 + 20 = -24. (22, -24).$

c $\sqrt{(0-22)^2 + (20-(-24))^2} = \sqrt{484+1936} = \sqrt{2420} = 49.2 \text{ km}$

Exercise 3C

1 **a** **i** $y - 9 = 2(x-3)$

ii $y = 2x + 3$

iii $2x - y + 3 = 0$

b **i** $y - 5 = \frac{1}{2}(x-6)$

ii $y = \frac{1}{2}x + 2$

iii $x - 2y + 4 = 0$

c **i** $y + 7 = -\frac{1}{3}(x-6)$

ii $y = -\frac{1}{3}x - 5$

iii $x + 3y + 15 = 0$

2 **a** $m = \frac{11-5}{5-2} = \frac{6}{3} = 2.$

i $y - 5 = 2(x-2)$

ii $y = 2x + 1$

iii $-2x + y - 1 = 0$

b $m = \frac{4-2}{0-2} = -1.$

i $y - 4 = -x$

ii $y = -x + 4$

iii $x + y - 4 = 0$

c $m = \frac{9-6}{3-2} = 3$

i $y - 6 = 3(x-2)$

ii $y = 3x$

iii $3x - y = 0$

d $m = \dfrac{-8+6}{1+2} = -\dfrac{2}{3}$

i $y + 6 = -\dfrac{2}{3}(x+2)$

ii $y = -\dfrac{2}{3}x - \dfrac{22}{3}$

iii $2x + 3x + 22 = 0$

3 **a** $3y = 2x - 7 \Rightarrow y = \dfrac{2}{3}x - \dfrac{7}{3}$, Gradient is $\dfrac{2}{3}$.

b $7y = -4x + 6 \Rightarrow y = -\dfrac{4}{7}x + \dfrac{6}{7}$, Gradient is $-\dfrac{4}{7}$.

c $by = -ax - d \Rightarrow y = -\dfrac{a}{b}x - \dfrac{d}{b}$, Gradient is $-\dfrac{a}{b}$.

4 **a** $m = \dfrac{5-2}{1-3} = \dfrac{3}{-2} \Rightarrow y = -\dfrac{3}{2}x + c \Rightarrow 5 = -\dfrac{3}{2}\times 1 + c \Rightarrow c = \dfrac{13}{2}$.

$y = -\dfrac{3}{2}x + \dfrac{13}{2}$.

b $2y = -3x + 13 \Rightarrow 3x + 2y - 13 = 0$.

c The height of the garden is $y = c = \dfrac{13}{2}$ m.

The width of the garden satisfies $0 = -3x + 13$ so is $x = \dfrac{13}{3}$ m.

The area of grass he needs to buy is $\dfrac{1}{2}\left(\dfrac{13}{2}\times\dfrac{13}{3}\right) = \dfrac{169}{12} = 14.1\,\text{m}^2$.

Exercise 3D

1 **a** **i** $2x + y = 8 \Rightarrow 4x + 2y = 16$. Add this to the other equation to get

$7x = 49 \Rightarrow x = 7 \Rightarrow \quad y = 8 - 14 = -6$.

Solution is $x = 7, y = 6$.

ii $2x + 10y = 3 \Rightarrow 6x + 30y = 9$. $\qquad 3x + 15y = 4.5 \Rightarrow 6x + 30y = 9$.

Two lines are the same.

iii Substitute the first equation into the second to get $4x - 2x - 1 = 5 \Rightarrow 2x = 6 \Rightarrow x = 3$.

So $y = 2\times 3 + 1 = 7$.

Solution is $x = 3, y = 7$.

iv Substitute the second equation into the first: $2x - 11 = 3x - 12 \Rightarrow x = 1 \Rightarrow y = 2\times 1 - 9$

Solution is $x = 1, y = -7$.

b **i** $x+3y=1 \Rightarrow 5x+15y=5.$

Subtract this from second equation to get $y=3.$

$x=1-3y \Rightarrow x=-8.$

Solution is $x=-8, y=3.$

ii $3x+2y=4 \Rightarrow 6x+4y=8.$ Add this to second equation to get

$11x=28.6 \Rightarrow x=2.6.$

$3\times 2.6+2y=4 \Rightarrow y=-1.9.$

Solution is $x=2.6,\ \ y=-1.9.$

2 **a** $y=\frac{1}{2}\times 50-100=25-100=-75.$ So Bernard is on the road.

b Subtract the first equation from the second:

$\frac{3}{2}x-510=0 \Rightarrow x=340.\ \ y=410-340=70.$

Roads intersect at

$(340,70)$.

c **i** The distance Alison has to walk is $\sqrt{(340-0)^2+(70-410)^2}=480.8\ldots\text{m}$.

The distance Bernard has to walk is $\sqrt{(340-50)^2+(70-(-75))^2}=324.2\ldots\text{m}.$

So Bernard arrives first.

ii Alison needs to travel an extra $480.8\ldots-324.22\ldots\approx 156.6\text{m}$, which will take $\frac{156.6}{4000}=0.03915\ldots$ hours or 2.349 minutes or 141 seconds.

3 **a** $\frac{5}{20}\times 100=25\%.$

b The way down is steeper.

c **i** $m=0.1.\ \ c=0.$

$y=0.1x.$

ii $m=-0.15.$ Passes through the point $(2.45,0).$

$0=-0.15\times 2.45+c.\ \ \ \ c=0.3675.$

$y=-0.15x+0.3675$

d Substitute the first equation into the second:

$0.1x=-0.15x+0.3675 \Rightarrow x=1.47.$

Height of the hill is $y=0.1\times 1.47=0.147$ km or 147m

e The distance up is $\sqrt{(1.47)^2 + 0.147^2} = 1.477\,\text{km}$.

The distance down is $\sqrt{(2.45 - 1.47)^2 + 0.147^2} = 0.9910\,\text{km}$.

So the total distance is $2.47\,\text{km}$.

4 **a** Pick any point on the straight line (except the one on the *x* axis). The straight line defines a right-angled triangle, with the chosen point as one of the vertices. The straight line is the hypotenuse, the *x*-axis contains the adjacent side with respect to α and the line vertically down from the chosen point forms the opposite side. Then the gradient of the line is $\frac{\text{change of y coordinate}}{\text{change in x coordinate}} = \frac{\text{length of opposite side}}{\text{length of adjacent side}} = \tan\alpha$.

b $m = \tan 4 = 0.069927$

$580 = 0.069927 \times 7500 + c$. $\quad c = 55.549$.

$y = 0.0699x + 55.5$

c $y = c = 55.5$ m.

d $0 = 0.0699x + 55.5 \Rightarrow x = -794$.

The aircraft lands $794 - 700 = 94\,\text{m}$ from the start of the runway.

Exercise 3E

1 **a** $\frac{1}{2}$ **b** -3 **c** -2 **d** $\frac{7}{6}$

2 **a** Gradient of $bx - ay = d_1$ is $m_1 = \frac{b}{a}$. Gradient of $ac + by = d_2$ is $m_2 = -\frac{a}{b}$.

$m_1 m_2 = \frac{b}{a} \times \left(-\frac{a}{b}\right) = -1$.

b $x + 2y - 10 = 0$

So, $b = 1, a = -2, d_1 = 10$

Perpendicular line:

$$\begin{aligned} ax + by &= d_2 \\ -2x + y &= d_2 \\ -2 \times 2 + 5 &= d_2 \\ d_2 &= 1 \end{aligned}$$

Hence, $-2x + y = 1$ or $2x - y + 1 = 0$

c $3x - 2y = 7$

So, $b = 3, a = 2, d_1 = 7$

Perpendicular line:

$$\begin{aligned} ax+by&=d_2 \\ 2x+3y&=d_2 \\ 2\times 6+3\times 5&=d_2 \\ d_2&=27 \end{aligned}$$

Hence, $2x+3y=27$ or $2x+3y-27=0$

3 a If you have any other point, B, on the line then this point, A and the point of intersection between the line and its perpendicular line through A form a right-angled triangle. But, the hypotenuse of this triangle is the line joining A and B and so is longer than the length of the perpendicular line.

b $4x+3y=d_1$. $d_1=4\times 5+3\times(-7)=-1$. $4x+3y+1=0$.

c $3x-4y+7=0\Rightarrow 9x-12y+21=0$. $4x+3y+1=0\Rightarrow 16x+12y+4=0$

$25x+25=0\Rightarrow x=-1$. $3\times -1-4\times y+7=0\Rightarrow y=1$.

Point of intersection is $(-1,1)$.

d Shortest distance of A from l is $\sqrt{(5-(-1))^2+(-7-1)^2}=\sqrt{36+64}=10$

4 a Perpendicular line is $5x-3y=d_1$. $d_1=5\times 2-3\times 4=-2$. $5x-3y+2=0$.

Finding the point of intersection:

$5x-3y+2=0\Rightarrow 25x-15y+10=0$

$3x+5y+8=0\Rightarrow 9x+15y+24=0$.

Adding these two equations:

$34x=-34\Rightarrow x=-1$.

$3\times -1+5y+8=0\Rightarrow y=-1$.

Point of intersection is $(-1,-1)$.

Shortest distance is $\sqrt{(2+1)^2+(4+1)^2}=\sqrt{9+25}=\sqrt{34}$.

b Perpendicular line: $y=-\frac{1}{3}x+c$. $-1=-\frac{1}{3}\times 5+c$. $c=\frac{2}{3}$. $y=-\frac{1}{3}x+\frac{2}{3}$

Finding the point of intersection: Take the equation of the perpendicular line away from the equation of the original line

$0=\frac{10}{3}x-\frac{8}{3}\Rightarrow x=\frac{4}{5}$.

$y=3\times\frac{4}{5}-2=\frac{2}{5}$.

$\left(\frac{4}{5},\frac{2}{5}\right)$.

Shortest distance is $\sqrt{\left(5-\frac{4}{5}\right)^2+\left(-1-\frac{2}{5}\right)^2}=\sqrt{19.6}=4.43$.

5 The gradient of AB is $\frac{18-16}{12-17}=-\frac{2}{5}$.

The midpoint of this line is $\left(\frac{12+17}{2},\frac{18+16}{2}\right)=(14.5,17)$.

Perpendicular bisector is
$y=2.5x+c$. $17=2.5\times14.5+c$. $c=-19.25$. $y=2.5x-19.25$

Line to the northeast passing through A is $y=x+c$. $18=12+c$. $c=6$. $y=x+6$.

Point of intersection: $2.5x-19.25=x+6\Rightarrow x=\frac{101}{6}=16.83$.

$y=x+6=22.83$.

The ship is at $(16.8,22.8)$ when it is north east of A.

6 **a** We need to minimise AS + SB. The minimum value is when the three points are collinear. Hence, S lines on the line between A and B, and also on $y=x+10$. (Note that S is not necessarily between A and B.)

The gradient of the line between the two towns is $m=\frac{24-16}{17-1}=\frac{8}{16}=\frac{1}{2}$.

So the equation of the line between the two towns is $y-16=\frac{1}{2}(x-1)$.

To find the point of intersection, substitute the track equation in the equation above

$x+10-16=\frac{1}{2}(x-1)\Rightarrow x=11$.

$y=x+10=21$.

Station should be built at $(11,21)$.

Total distance is $\sqrt{(17-1)^2+(24-16)^2}\approx17.9\text{km}$.

b To be the same distance from each town the station must lie on the perpendicular bisector of [AB]. Finding the perpendicular bisector: Midpoint of [AB] is $\left(\frac{1+17}{2},\frac{16+24}{2}\right)=(9,20)$.

The gradient of the perpendicular bisector is $-\frac{2}{1}=-2$.

The equation of the perpendicular bisector is

$y-20=-2(x-9)$.

Point of intersection: Substitute the equation of the rail track into the equation above

$x+10-20=-2(x-9)\Rightarrow3x=28\Rightarrow x=9\frac{1}{3}$.

$y = 9\frac{1}{3} + 10 = 19\frac{1}{3}$.

Station should be built at $\left(9\frac{1}{3}, 19\frac{1}{3}\right)$

Total distance is $2 \times \sqrt{\left(9\frac{1}{3} - 1\right)^2 + \left(19\frac{1}{3} - 16\right)^2} \approx 18.0$ km.

Exercise 3F

1 **a** Finding the perpendicular bisectors:

$m_1 = -\frac{2-1}{3-1} = -\frac{1}{2}$.

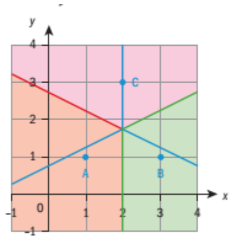
Midpoint of AC is $\left(\frac{2+1}{2}, \frac{3+1}{2}\right) = (1.5, 2)$.

$y - 2 = -\frac{1}{2}\left(x - \frac{3}{2}\right)$.

$m_2 = -\frac{2-3}{3-1} = \frac{1}{2}$.

Midpoint of BC is $\left(\frac{2+3}{2}, \frac{3+1}{2}\right) = (2.5, 2)$.

$y - 2 = \frac{1}{2}\left(x - \frac{5}{2}\right)$



b Finding the perpendicular bisectors:

$m_1 = -\frac{(5-1)}{5-1} = -1$.

Midpoint of AC is $\left(\frac{5+1}{2}, \frac{5+1}{2}\right) = (3, 3)$.

$y - 3 = -(x - 3)$.

$m_2 = -\frac{3-1}{5-1} = -\frac{1}{2}$.

Midpoint of AD is $\left(\frac{3+1}{2}, \frac{5+1}{2}\right) = (2, 3)$.

$y-3=-\frac{1}{2}(x-2)$.

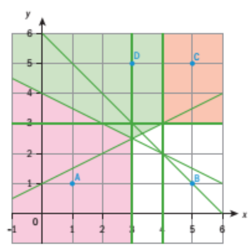
Perpendicular bisector between A and B is $y=3$.

$m_3=-\frac{3-5}{5-1}=\frac{1}{2}$.

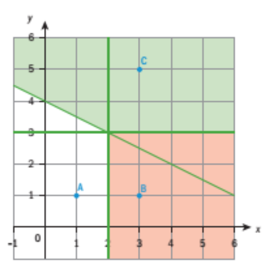
Midpoint of BD is $\left(\frac{3+5}{2},\frac{5+1}{2}\right)=(4,3)$.

$y-3=\frac{1}{2}(x-4)$.

Perpendicular bisector of DC is $x=4$



2 a

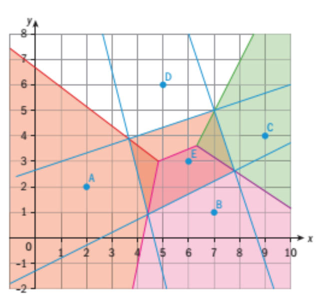


b The reading at point (1,4) would be 21ºC.

3 a Gradient between E and D is: $m=\frac{3-6}{6-5}=-3$.

Midpoint is $\left(\frac{6+5}{2},\frac{3+6}{2}\right)=(5.5,4.5)$.

Perpendicular bisector: $y-4.5=\frac{1}{3}(x-5.5)$ or $x-3y=8$

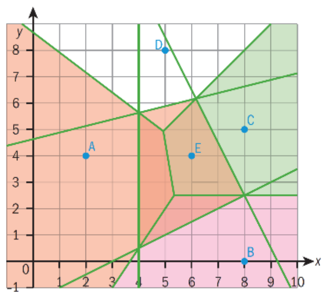


b Perpendicular bisector between A and E is x = 4.

Gradient between B and E is $m = \dfrac{0-4}{8-6} = -2$.

Midpoint is $\left(\dfrac{8+6}{2}, \dfrac{0+4}{2}\right) = (7, 2)$

Perpendicular bisector: $y - 2 = \dfrac{1}{2}(x - 7)$ or $x - 2y = 3$



i 19mm

ii 21mm

c **i** 11mm

ii 12mm

Exercise 3G

1 **a** This point will be a vertex in a Voronoi diagram, as otherwise the point will be in a cell, and can therefore be moved further away from the corresponding school by moving the point to a vertex of that cell. The only vertex of the Voronoi diagram in this example is the meeting point of the three perpendicular bisectors of [AB], [AC] and [BC].

b Perpendicular bisector of [AB]: $m = -\dfrac{6-1}{4-3} = -5$.

Midpoint is $\left(\dfrac{1+6}{2}, \dfrac{3+4}{2}\right) = (3.5, 3.5)$.

$$y - 3.5 = -5(x - 3.5)$$

$$y = -5x + 21$$

Perpendicular bisector of [BC]: $y = 2.5$.

c $2.5 - 3.5 = -5(x - 3.5)$

$-1 = -5x + 17.5 \Rightarrow x = 3.7$.

New school should be built at $(3.7, 2.5)$.

d This new school is the same distance from the three original schools:

$$d = \sqrt{(3.7-1)^2 + (2.5-3)^2}$$

$$= 2.75\text{ km}$$

2 **a** **i** $m = -\dfrac{40-20}{10-30} = 1.$

Midpoint is $\left(\dfrac{20+40}{2}, \dfrac{30+10}{2}\right) = (30, 20).$

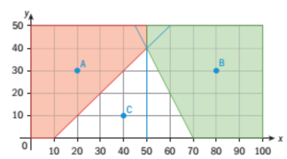
$y = x - 10.$

ii $m = -\dfrac{80-40}{30-10} = -2.$

Midpoint is $\left(\dfrac{80+40}{2}, \dfrac{30+10}{2}\right) = (60, 20).$

$y = -2x + 140.$

b



c **i** Area of the fairground that will go to stand C is

$\dfrac{1}{2}(60 \times 40) = 1200.$

Total area of fairground is 5000.

So proportion that will go to stand C is

$\dfrac{1200}{5000} = 0.24$

ii Area of the fairground that will go to stand A is

$(10 \times 50) + (10 \times 40) + \dfrac{1}{2}(40 \times 40) = \quad 500 + 400 + 800 = 1700.$

So proportion that will go to stand A is

$\dfrac{17}{50} = 0.34$

d **i** The stand should be built at the intersection of all three perpendicular bisectors. That is, it is the solution to $y = x - 10$ and $y = -2x + 140$. Thus,

$$x - 10 = -2x + 140$$
$$3x = 150$$
$$x = 50$$
$$y = 40$$

New stand is at $(50, 40)$.

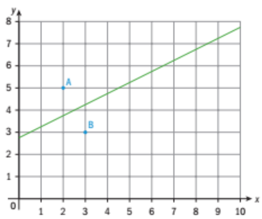
ii $d = \sqrt{(50-40)^2 + (40-10)^2} = 31.6.$ The distance from all three other stands is 31.6 m

3 **a** $m = -\frac{3-2}{3-5} = \frac{1}{2}.$

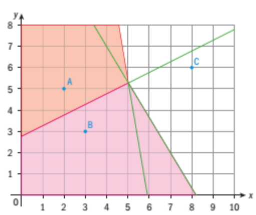
midpoint is $\left(\frac{2+3}{2}, \frac{3+5}{2}\right) = (2.5, 4).$

$y = \frac{1}{2}x + 2.75.$

b



c

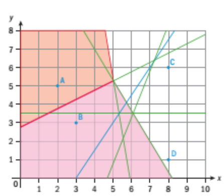


d **i** BD: $m = -\frac{8-3}{1-3} = \frac{5}{2}$

midpoint is $\left(\frac{3+8}{2}, \frac{3+1}{2}\right) = (5.5, 2).$

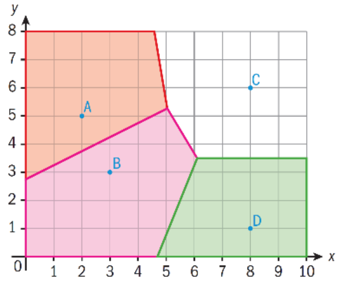
$y - 2 = \frac{5}{2}(x - 5.5)$ or $y = \frac{5}{2}x - 11.75$

CD: $y = 3.5$



ii B and C are closer to D than they are to A.

iii



e $y = 3.5$ meets $y = -\frac{5}{3}x + \frac{41}{3}$ when $3.5 = -\frac{5}{3}x + \frac{41}{3} \Rightarrow x = \frac{41 - 10.5}{5} = 6.1$.

$y = -\frac{5}{3}x + \frac{41}{3}$ meets $y = -6x + 35.5$ when $-6x + 35.5 = -\frac{5}{3}x + \frac{41}{3} \Rightarrow 13x = 65.5 \Rightarrow x = 5.038$.

$y = -\frac{5}{3}(5.038) + \frac{41}{3} = 5.27$.

The vertices are at $(6.1, 3.5)$ and $(5.04, 5.27)$.

f Area C is bounded by (5.04,5.27), (6.1,3.5), (10,3.5), (10,8) and a point given by $y = -6x + 35.5$ when y = 8. This point is (4.58,8).

Area of upper trapezium = $\frac{1}{2}\left[(10-4.58)+(10-5.04)\right]\times(8-5.27)=14.17$

Area of lower trapezium = $\frac{1}{2}\left[(10-5.04)+(10-6.1)\right]\times(5.27-3.5)=7.84$

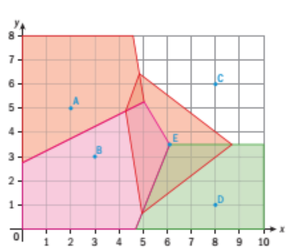
Area of C = 14.2+7.8=22.0

Total area is 80. So percentage area is $\frac{22}{80}\times 100 = 28\%$

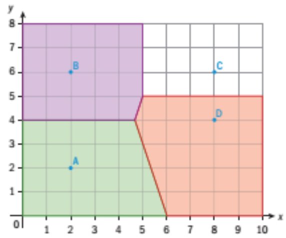
g Distance from the vertex at (6.1,3.5) to the vertex D is $\sqrt{(8-6.1)^2 + (1-3.5)^2} = 3.14$.

Distance from the vertex at (5.038,5.27) to A is $\sqrt{(5.04-2)^2 + (5.27-5)^2} = 3.05$. So the new school should be built at (6.1,3.5).

h



4 a



b Finding perpendicular bisector of AD: $m = -\dfrac{8-2}{4-2} = -3$.

$\left(\dfrac{8+2}{2}, \dfrac{4+2}{2}\right) = (5,3)$.

$y = -3x + 18$.

This meets y = 4 when $4 = -3x + 18 \Rightarrow x = \dfrac{14}{3} = 4.67$

$(4.67, 4)$.

c i Trapezium: $\dfrac{1}{2} \times 4 \times \left(6 + \dfrac{14}{3}\right) = \dfrac{64}{3} = 21.3$ or 213000 miles^2

ii Rectangle + trapezium: $(5 \times 3) + \dfrac{1}{2} \times 1 \times \left(5 + \dfrac{14}{3}\right) = 19.8$ or 198000 miles^2

iii Rectangle: $3 \times 5 = 15$ or 150000 miles^2

iv $80 - 15 - 19.8 - 21.3 = 23.9$ or 239000 miles^2

d Not possible to support the other province.

5 a i AB = 6, BC = 3, CD = 4 and DA = $\sqrt{(2-0)^2 + (5-2)^2} = 3.61$.

Total distance is $6 + 3 + 4 + 3.61 = 16.61$ or 166 km

ii Distance travelled while closest to A is $\dfrac{1}{2}(6 + 3.61) = 4.81$ or 48.1km.

So, proportion of journey spent closest to A is $\dfrac{48.1}{166} = 0.290$.

b i This is reasonable because (4,3.5) is a vertex in the Voronoi diagram and so it is an equal distance from three stations. The three stations are B, C and D.

ii $\dfrac{2.1 + 2.6 + 2.8}{3} = 2.5$.

c i Median=3.0. Lower quartile= 2.7. Upper Quartile= 3.1. Interquartile range=0.4

ii Outliers would be below $2.7 - 1.5 \times 0.4 = 2.1$ or above $3.1 + 1.5 \times 0.4 = 3.7$. So the reading of 2.1 is not an outlier because it is on the threshold not below it.

d All the readings, apart from the lowest one, are above the expected value the officer calculated. So, the readings back up the houseowner's claim.

Exercise 3H

1 **a** $\begin{pmatrix} 0 \\ -3 \end{pmatrix}, -3j$ **b** $\begin{pmatrix} -2 \\ -2 \end{pmatrix}, -2i - 2j$ **c** $\begin{pmatrix} 2 \\ 1 \end{pmatrix}, 2i + j$ **d** $\begin{pmatrix} 1 \\ -3 \end{pmatrix}, i - 3j$

e $\begin{pmatrix} 2 \\ 0 \end{pmatrix}, 2i$

2 **a** $\begin{pmatrix} -3 \\ 4 \end{pmatrix}$ **b** $7i + 4j$ **c** $\begin{pmatrix} -6 \\ 2 \end{pmatrix}$ **d** $5i - 2j$

3 **a** $\begin{pmatrix} 4 \\ 6 \end{pmatrix}$ **b** $\begin{pmatrix} 6 \\ 9 \end{pmatrix}$

c To multiply a vector by a scalar, you multiply each component of the vector by that scalar.

d $3i + 4j = 3\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 4\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}.$

4 **a** **i** $CD = \begin{pmatrix} 1 \\ 2 \end{pmatrix} = AB$

ii $FE = \begin{pmatrix} -1 \\ -2 \end{pmatrix} = -AB$

iii $GH = \begin{pmatrix} 2 \\ 4 \end{pmatrix} = 2AB$

iv $\overrightarrow{IJ} = \begin{pmatrix} -0.5 \\ -1 \end{pmatrix} = -\frac{1}{2}\overrightarrow{AB}$

b Parallel vectors are scalar multiples of each other.

5 **a** Parallel. **b** Not parallel. **c** Parallel **d** Not parallel

e Parallel **f** Not parallel **g** Parallel

6 **a** **i** $4p + 6 = -2p \Rightarrow p = -1.$ $6 - 2q = 2 \Rightarrow q = 2$

ii $3p + 2q = 7.$ $-2q + p = 1 \Rightarrow 4p = 8 \Rightarrow p = 2.$ $-2q = 1 - 2 = -1 \Rightarrow q = \frac{1}{2}.$

b **i** $\begin{pmatrix} p+1 \\ 2p \end{pmatrix} = k\begin{pmatrix} 4 \\ 1 \end{pmatrix} \Rightarrow p + 1 = 4k.$ $2p = k \Rightarrow$ $p + 1 = 8p \Rightarrow p = \frac{1}{7}.$

ii $\begin{pmatrix} 2q-3 \\ q+6 \end{pmatrix} = k\begin{pmatrix} -3 \\ 1 \end{pmatrix} \Rightarrow 2q - 3 = -3k,$ $q + 6 = k \Rightarrow$ $2q - 3 = -3q - 18 \Rightarrow 5q = -15 \Rightarrow q = -3.$

Exercise 3I

1 **a** **i** $a = \begin{pmatrix} 1 \\ 4 \end{pmatrix}.$

ii $AB = b - a = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

iii $AC = c - a = \begin{pmatrix} -2 \\ 2 \end{pmatrix}$

iv $CA = -AC = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$

b **i** $BD = BA + AC + CD$

ii $BD = -AB + AC + CD$

iii $BD = -\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} -2 \\ 2 \end{pmatrix} + \begin{pmatrix} 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

iv $d = OB + BD = b + BD = \begin{pmatrix} 2 \\ 5 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 7 \end{pmatrix}.$

2 **a** **i** $AC = AB + BC = \begin{pmatrix} 0 \\ 6 \end{pmatrix}$

ii $CA = \begin{pmatrix} 0 \\ -6 \end{pmatrix}$

b $DC = DA + AB + BC = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$

3 **a** $AB = b - a = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$. $DC = c - d = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$.

b Parallelogram – two opposite sides have the same direction and length.

c AD and BC

4 **a** $AC = \begin{pmatrix} 40 \\ -10 \end{pmatrix}$.

b $\begin{pmatrix} -40 \\ 10 \end{pmatrix}$

c $d = \sqrt{40^2 + 10^2} = \sqrt{1700} = 41.2\text{km}$

5 **a** Final displacement vector is $(5 - 2 + 4 + 6)\boldsymbol{i} + (1 + 4 + 2 + 4)\boldsymbol{j} = 13\boldsymbol{i} + 11\boldsymbol{j}.$

b Displacement vector which will take him back to the start is $-13\boldsymbol{i} - 11\boldsymbol{j}$.

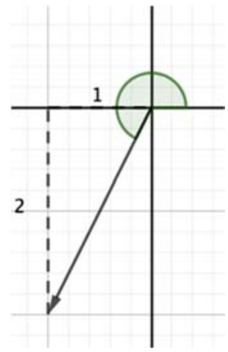
Exercise 3J

1 **a** **i** Resultant is $\begin{pmatrix} 4 \\ 4 \end{pmatrix}$. Magnitude is $\sqrt{4^2 + 4^2} = 5.66$

ii Angle is $\tan^{-1}\frac{4}{4} = 45°$

b **i** Resultant is $-i-2j$. Magnitude is $\sqrt{1^2+2^2}=2.24$

ii It is a good idea to always draw a diagram.



Angle is $180^\circ + \tan^{-1}\left(\frac{2}{1}\right)=243^o$

c **i** Resultant is $\begin{pmatrix}18\\7\end{pmatrix}$. Magnitude is $\sqrt{18^2+7^2}=19.3$

ii Angle is $\tan^{-1}\left(\frac{7}{18}\right)=21.3^o$

d **i** Resultant is $8i+j$. Magnitude is $\sqrt{8^2+1^2}=8.06$

ii Angle is $\tan^{-1}\left(\frac{1}{8}\right)=7.13^o$

2 **a** $\begin{vmatrix}48\\20\end{vmatrix}=\sqrt{48^2+20^2}=\sqrt{2704}=\sqrt{16\times169}=4\times\sqrt{169}=4\sqrt{12^2+5^2}=4\begin{vmatrix}12\\5\end{vmatrix}$

b **i** $\begin{vmatrix}18\\24\end{vmatrix}=6\begin{vmatrix}3\\4\end{vmatrix}=6\sqrt{3^2+4^2}=6\times5=30.$

ii $\begin{vmatrix}-30\\40\end{vmatrix}=10\begin{vmatrix}-3\\4\end{vmatrix}=10\times5=50.$

iii $\begin{vmatrix}28\\-21\end{vmatrix}=7\begin{vmatrix}4\\-3\end{vmatrix}=7\times5=35.$

3 **a** $\sqrt{3^2+4^2}=5.$

$\frac{8}{5}\begin{pmatrix}3\\4\end{pmatrix}=\frac{1}{5}\begin{pmatrix}24\\32\end{pmatrix}$

b $\sqrt{74-5^2}=\sqrt{49}=7.$

$\begin{pmatrix}5\\7\end{pmatrix}$

c $\sqrt{50}=\begin{vmatrix}k+1\\k-5\end{vmatrix}=\sqrt{(k+1)^2+(k-5)^2}=\sqrt{2k^2-8k+26}\Rightarrow 50=2k^2-8k+26.$

$\Rightarrow k^2 - 4k - 12 = 0 = (k-6)(k+2)$. Since $k > 0$, the vector is $\begin{pmatrix} 7 \\ 1 \end{pmatrix}$.

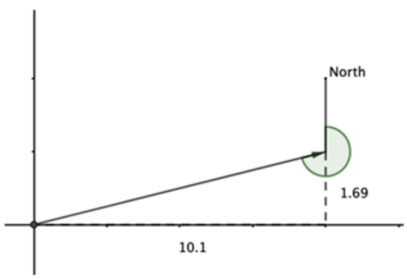
4 **a** North-east for 200 m: $k\begin{vmatrix} 1 \\ 1 \end{vmatrix} = \sqrt{2}k = 200 \Rightarrow k = 100\sqrt{2} = 141.4$.

Vector is: $\begin{pmatrix} 141 \\ 141 \end{pmatrix}$

West for 175 m: $\begin{pmatrix} -175 \\ 0 \end{pmatrix}$.

b $\begin{vmatrix} 141-175 \\ 141 \end{vmatrix} = \sqrt{21037} = 145$ km

5 $\begin{vmatrix} 4\sin(30)+3\sin(135)+4+2\sin(80) \\ 4\cos(30)+3\cos(135)+0+2\cos(80) \end{vmatrix} = \begin{vmatrix} 10.1 \\ 1.69 \end{vmatrix} = \sqrt{104.8661} = 10.2$ km.



Bearing is $\tan^{-1}\left(\dfrac{10.1}{1.69}\right) = 80.5^0$. So to travel back to the start has to travel at this bearing plus 180° so at a bearing of 261°.

Exercise 3K

1 **a** $\sqrt{8^2+4^2+1^2} = \sqrt{64+16+1} = \sqrt{81} = 9$.

b $\sqrt{2^2+1^2+2^2} = \sqrt{4+1+4} = \sqrt{9} = 3$.

c $3\sqrt{4^2+3^2} = 3\sqrt{16+9} = 3\times 5 = 15$.

d $5\sqrt{2^2+3^2+6^2} = 5\sqrt{4+9+36} = 5\sqrt{49} = 5\times 7 = 35$.

2 $\begin{pmatrix} -12 \\ -9 \\ -6 \end{pmatrix}$.

3 **a** To get from A to B, travel down the vector $AB = b - a$ so to travel from A to the midpoint of [AB] travel down $AM = \frac{1}{2}AB = \frac{1}{2}(b-a)$. Now $OM = OA + AM = a + \frac{1}{2}(b-a)$

$OM = \frac{1}{2}(a+b)$.

b $PQ = q - p = \begin{pmatrix} -2 \\ -3 \\ -1 \end{pmatrix}$. $\quad QR = r - q = \begin{pmatrix} 3 \\ 4 \\ -6 \end{pmatrix}$.

c $PR = PQ + QR = \begin{pmatrix} 1 \\ 1 \\ -7 \end{pmatrix}$.

d $PS = QR$. So S has the coordinates $(1+3, 3+4, 6+-6) = (4, 7, 0)$.

e $\frac{1}{2}PR = \begin{pmatrix} 0.5 \\ 0.5 \\ -3.5 \end{pmatrix}$

f $\frac{1}{2}QS = \frac{1}{2}(s - q) = \frac{1}{2}\begin{pmatrix} 5 \\ 7 \\ -5 \end{pmatrix} = \begin{pmatrix} 2.5 \\ 3.5 \\ -2.5 \end{pmatrix}$.

g The midpoint of the diagonal PR is $(1+0.5, 3+0.5, 6-3.5) = (1.5, 3.5, 2.5)$. Midpoint of the diagonal QS is $(-1+2.5, 0+3.5, 5-2.5) = (1.5, 3.5, 2.5)$. So the two diagonals bisect.

Exercise 3L

1 a $\frac{a \cdot b}{|a||b|} = \frac{2\times3 - 1\times1 + 4\times2}{\sqrt{(2^2+1^2+4^2)(3^2+1^2+2^2)}} = \frac{13}{\sqrt{294}}$. $\quad \theta = \cos^{-1}\left(\frac{13}{\sqrt{294}}\right) = 40.7°$.

b $\frac{a \cdot b}{|a||b|} = \frac{2\times-2 + 0\times1 + 1\times-1}{\sqrt{(2^2+0^2+1^2)(2^2+1^2+-1^2)}} = \frac{-5}{\sqrt{30}}$. $\quad \theta = \cos^{-1}\left(\frac{-5}{\sqrt{30}}\right) = 156°$.

c $\frac{a \cdot b}{|a||b|} = \frac{2\times3 + 1\times2 - 1\times0}{\sqrt{(2^2+1^2+1^2)(3^2+2^2)}} = \frac{8}{\sqrt{78}}$. $\quad \theta = \cos^{-1}\left(\frac{8}{\sqrt{78}}\right) = 25.1°$.

d $\frac{a \cdot b}{|a||b|} = \frac{2\times3 - 1\times2 - 2\times-5}{\sqrt{(2^2+1^2+2^2)(3^2+2^2+5^2)}} = \frac{14}{\sqrt{342}}$. $\quad \theta = \cos^{-1}\left(\frac{14}{\sqrt{342}}\right) = 40.8°$.

e $\frac{a \cdot b}{|a||b|} = \frac{2\times4 + 3\times6}{\sqrt{(2^2+3^2)(4^2+6^2)}} = \frac{26}{\sqrt{676}}$. $\quad \theta = \cos^{-1}\left(\frac{26}{\sqrt{676}}\right) = 0°$.

f $\begin{pmatrix} -4 \\ 0 \\ -2 \end{pmatrix} = -2\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$. So the vectors are parallel but in opposite directions. So the angle is 180°.

2 a i AC and AB.

ii BC and BA

b $x = AB = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}, \quad y = BC = \begin{pmatrix} 1 \\ 1 \\ -7 \end{pmatrix}, \quad z = AC = \begin{pmatrix} 0 \\ 1 \\ -5 \end{pmatrix}.$

Angle at A: $\frac{x \cdot z}{|x||z|} = \frac{-1 \times 0 + 0 \times 1 + 2 \times -5}{\sqrt{(1^2 + 0^2 + 2^2)(0^2 + 1^2 + 5^2)}} = \frac{-10}{\sqrt{130}}. \quad A = \cos^{-1}\left(\frac{-10}{\sqrt{130}}\right) = 151^\circ.$

Angle at B: $\frac{y \cdot -x}{|y||x|} = \frac{1 \times 1 + 1 \times 0 - 7 \times -2}{\sqrt{(1^2 + 1^2 + 7^2)(1^2 + 0^2 + 2^2)}} = \frac{15}{\sqrt{255}}. \quad B = \cos^{-1}\left(\frac{15}{\sqrt{255}}\right) = 20.1^\circ$

$C = 180 - 151 - 20.1 = 8.9^\circ.$

c The longest side is the one opposite angle A, so is side BC. This length is the length of

$BC = \sqrt{1^2 + 1^2 + 7^2} = \sqrt{51} = 7.14$

3 a $a \cdot b = -6p + 2 - 2p = 0 \Rightarrow 8p = 2 \Rightarrow p = \frac{1}{4}.$

b

$$\begin{aligned} a \cdot b &= p(p-1) - 2p - 4 = 0 \\ p^2 - p - 2p - 4 &= 0 \\ p^2 - 3p - 4 &= 0 \\ (p-4)(p+1) &= 0 \\ p &= -1,\ 4. \end{aligned}$$

4 a

$$x = AC = \begin{pmatrix} 1 \\ 0 \\ k-2 \end{pmatrix}, \quad y = BC = \begin{pmatrix} 0 \\ -1 \\ k+1 \end{pmatrix}$$

$$x \cdot y = (k-2)(k+1) = 0$$

Hence, $k = -1$ or 2.

b When k = −1: $AC = \sqrt{(1-2)^2 + (3-3)^2 + (2-(-1))^2} = \sqrt{10}.$

$BC = \sqrt{(2-2)^2 + (4-3)^2 + (-1-(-1))^2} = 1.$

Area of the triangle is $\frac{\sqrt{10}}{2}.$

When k = 2: $AC = \sqrt{(1-2)^2 + (3-3)^2 + (2-2)^2} = 1, \ BC = \sqrt{(2-2)^2 + (4-3)^2 + (-1-2)^2} = \sqrt{10}.$

Area of the triangle is $\frac{\sqrt{10}}{2}.$

5 Consider a unit cube that has one vertex at the origin and has one of its edges along the x-axis, one along the y-axis and the other along the z-axis. The vertex at the origin is (0, 0, 0) and the diagonally opposite vertex is (1, 1, 1) so the diagonal is the vector

$$a = \begin{pmatrix}1\\1\\1\end{pmatrix}.$$

The diagonal between the vertices (0,0,1) and (1,1,0) is

$$b = \begin{pmatrix}0-1\\0-1\\1-0\end{pmatrix} = \begin{pmatrix}-1\\-1\\1\end{pmatrix}$$

$\dfrac{a\cdot b}{|a||b|} = \dfrac{1\times -1 + 1\times -1 + 1\times 1}{\sqrt{(1^2+1^2+1^2)((-1)^2+(-1)^2+1^2)}} = -\dfrac{1}{3}.$ $\theta = \cos^{-1}\left(-\dfrac{1}{3}\right) = 109.47^o$. So the acute angle between two diagonals of a cube is $180 - 109.47 = 70.5^o$.

Exercise 3M

1 a $c = a\times b = \begin{pmatrix}(1\times 2)-(3\times -1)\\(3\times 1)-(2\times 2)\\(2\times -1)-(1\times 1)\end{pmatrix} = \begin{pmatrix}5\\-1\\-3\end{pmatrix}.$

b $a\cdot c = (2\times 5) + (1\times -1) + (3\times -3) = 10 - 1 - 9 = 0.$

$b\cdot c = (1\times 5) + (-1\times -1) + (2\times -3) = 5 + 1 - 6 = 0.$

2 a $a\times b = \begin{pmatrix}-1\times 2 + 1\times 4\\4\times 3 - 2\times 2\\2\times -1 - 3\times -1\end{pmatrix} = \begin{pmatrix}2\\8\\1\end{pmatrix}.$

b $a\times b = \begin{pmatrix}0\times -1 - 1\times 3\\1\times -2 - 2\times -1\\2\times 3 - 0\times -2\end{pmatrix} = \begin{pmatrix}-3\\0\\6\end{pmatrix}.$

c $a\times b = (1\times 0 - -1\times 2)i + (-1\times 3 - 2\times 0)j + (2\times 2 - 1\times 3)k = 2i - 3j + k$

d $a\times b = (-1\times -5 - -2\times 2)i + (-2\times 3 - 2\times -5)j + (2\times 2 - -1\times 3)k = 9i + 4j + 7k$

3 a $BC = AC - AB = \begin{pmatrix}-4\\1\\-2\end{pmatrix}.$

b $\text{area} = \dfrac{1}{2}|AB\times AC| = \dfrac{1}{2}\begin{vmatrix}0\times -1 - 1\times 1\\1\times -2 - 2\times -1\\2\times 1 - 0\times -2\end{vmatrix} = \dfrac{1}{2}\begin{vmatrix}-1\\0\\2\end{vmatrix} = \dfrac{1}{2}\sqrt{1^2+2^2} = \dfrac{\sqrt{5}}{2}.$

4 $AB = \begin{pmatrix}-1\\0\\2\end{pmatrix}$, $AC = \begin{pmatrix}0\\1\\-5\end{pmatrix}$. Area of ABC is $\dfrac{1}{2}|AB\times AC| = \dfrac{1}{2}\begin{vmatrix}0\times -5 - 2\times 1\\2\times 0 + 1\times -5\\-1\times 1 - 0\times 0\end{vmatrix} = \dfrac{1}{2}\begin{vmatrix}-2\\-5\\-1\end{vmatrix} = \dfrac{\sqrt{30}}{2}.$

5 a $AB = \begin{pmatrix}0\\0\\8\end{pmatrix}$, $d = c - AB = \begin{pmatrix}4\\3\\-3\end{pmatrix}.$

b Area is $\left|AB \times AD\right| = \left|\begin{pmatrix}0\\0\\8\end{pmatrix} \times \begin{pmatrix}3\\1\\0\end{pmatrix}\right| = \begin{vmatrix}0\times 0-8\times 1\\8\times 3-0\times 0\\0\times 1-0\times 3\end{vmatrix} = \begin{vmatrix}-8\\24\\0\end{vmatrix} = 8\begin{vmatrix}-1\\3\\0\end{vmatrix} = 8\sqrt{10}.$

c $AB \cdot AD = 0\times 3+0\times 1+8\times 0=0.$

6 $AB = \begin{pmatrix}-1\\-3\\1\end{pmatrix}$, $AD = \begin{pmatrix}6\\-2\\0\end{pmatrix}$. $AB\cdot AD = -1\times 6+\ -3\times -2+0\times -1 = 6-6=0.$ So AB and AD are perpendicular sides of the roof. So the total area is $|AB||AD| = \sqrt{(1^2+3^2+1^2)}\sqrt{6^2+2^2}$

$= \sqrt{11}\times\sqrt{40} = 21.0\ \text{m}^2.$

7 The vector corresponding to the side of the base between the two given adjacent corners is $a = \begin{pmatrix}3\\0\\-2\end{pmatrix}$. So the area of the base is $|a|^2 = a\cdot a = 3\times 3+0\times 0+-2\times -2 = 13.$ The vector corresponding to one of the edges from the vertex to the base is given by $b = \begin{pmatrix}1\\0\\-3\end{pmatrix}$. So the area of one of the triangular sides is $\frac{1}{2}|a\times b| = \frac{1}{2}\begin{vmatrix}0\times -3+2\times 0\\-2\times 1-3\times -3\\3\times 0-0\times 1\end{vmatrix} = \frac{1}{2}\begin{vmatrix}0\\7\\0\end{vmatrix} = \frac{7}{2}$. So the total area is

$13+4\times\frac{7}{2} = 13+14=27.$

Exercise 3N

1 **a** $r = \begin{pmatrix}2\\4\end{pmatrix}+t\begin{pmatrix}2-3\\4-1\end{pmatrix} = \begin{pmatrix}2\\4\end{pmatrix}+t\begin{pmatrix}-1\\3\end{pmatrix}.$ **b** $r = \begin{pmatrix}2\\1\\-1\end{pmatrix}+t\begin{pmatrix}4-2\\2-1\\0--1\end{pmatrix} = \begin{pmatrix}2\\1\\-1\end{pmatrix}+t\begin{pmatrix}2\\1\\1\end{pmatrix}.$

2 **a** $2+8t=-2 \Rightarrow t = -\frac{1}{2}.$ $a = 1-\frac{1}{2}\times 1 = \frac{1}{2},$ $b = 1-\frac{1}{2}\times -2 = 2.$

b $0=1+t \Rightarrow t=-1.$ $(1,2,1)-(1,8,-2) = (0,-6,3).$

3 **a** $2=1+t \Rightarrow t=1.$ $s = 0+1\times 3 = 3,$ $p-1 = 2+1\times 1 \Rightarrow p=4.$

b **i** $3+t=1 \Rightarrow t=-2,$ $(3,1,2)-2(1,2,-3) = (1,-3,8).$

ii $AB = (5--2)\begin{pmatrix}1\\2\\-3\end{pmatrix} = \begin{pmatrix}7\\14\\-21\end{pmatrix}.$

4 **a** $4-t=2 \Rightarrow t=2,$ $(4,-5,1)+2(-1,3,1) = (2,1,3).$

$4+2s=2 \Rightarrow s=-1,$ $(4,2,0)-(2,1,-3) = (2,1,3).$

b $\dfrac{\begin{pmatrix}-1\\3\\1\end{pmatrix}\cdot\begin{pmatrix}2\\1\\-3\end{pmatrix}}{\begin{vmatrix}-1\\3\\1\end{vmatrix}\begin{vmatrix}2\\1\\-3\end{vmatrix}}=\dfrac{-1\times 2+3\times 1+1\times -3}{\sqrt{\left(1^2+3^2+1^2\right)\left(2^2+1^2+3^2\right)}}=-\dfrac{2}{\sqrt{11\times 14}}$. $\quad\theta=\cos^{-1}\left(-\dfrac{2}{\sqrt{154}}\right)=99.27$. So acute angle is $180-99.27=80.7^{\circ}$.

5 a l_1 and l_2 are parallel as the direction vectors for each line are scalar multiples of each other.

b $3-t=1\Rightarrow t=2,\quad (3,5,2)+2(-1,2,1)=(1,9,4)$

$3+2s=1\Rightarrow t=-1,\quad (3,5,2)-(2,-4,-2)=(1,9,4)$

c This tells us that the two lines are the same line.

6 a $3+s=1\Rightarrow s=-2,\quad (3,1,2)-2(1,3,-2)=(1,-5,6)$.

b $AB=\begin{pmatrix}4\\2\\5\end{pmatrix},\quad AB\cdot\begin{pmatrix}1\\3\\-2\end{pmatrix}=1\times 4+2\times 3+5\times -2=0.$

7 a $(2+t,3t,1+2t)$.

b $AP=\begin{pmatrix}1+t\\3t-2\\3+2t\end{pmatrix}$.

c

$$AP\cdot\begin{pmatrix}1\\3\\2\end{pmatrix}=0$$

$$\begin{aligned}AP\cdot\begin{pmatrix}1\\3\\2\end{pmatrix}&=1(1+t)+3(3t-2)+2(3+2t)\\&=1+t+9t-6+6+4t\\&=14t+1\Rightarrow t=-\frac{1}{14}.\end{aligned}$$

d Point on l_1 that is closest to A is the point with

$t=-\dfrac{1}{14}$: $\quad (2,0,1)-\dfrac{1}{14}(1,3,2)=\left(\dfrac{27}{14},-\dfrac{3}{14},\dfrac{6}{7}\right)$.

e Shortest distance from A to l_1 is $\sqrt{\left(\dfrac{13}{14}\right)^2+\left(\dfrac{31}{14}\right)^2+\left(\dfrac{20}{7}\right)^2}=\dfrac{1}{14}\sqrt{2730}=3.73$

Exercise 3O

1 a i $v=\dfrac{1}{2}\left((4-2)i+(5-1)j\right)=i+2j$

ii $|v| = \sqrt{1^2 + 2^2} = \sqrt{5}.$

b **i** $v = \dfrac{1}{4}\begin{pmatrix} 1-2 \\ 1--1 \end{pmatrix} = \begin{pmatrix} -0.25 \\ 0.5 \end{pmatrix}.$

ii $|v| = \sqrt{0.25^2 + 0.5^2} = 0.559$

c **i** $v = \dfrac{1}{2}\big((1-3)i + (5-1)j + (1--1)k\big) = -i + 2j + k.$

ii $|v| = \sqrt{1^2 + 2^2 + 1^2} = \sqrt{6}.$

d **i** $v = \dfrac{1}{4}\begin{pmatrix} 1-1 \\ 4-0 \\ -3-1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$

ii $|v| = \sqrt{0^2 + 1^2 + 1^2} = \sqrt{2}.$

2 **a** $10\begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} -10 \\ 0 \end{pmatrix}.$

b $7.5\dfrac{\begin{pmatrix} -3 \\ 4 \end{pmatrix}}{\begin{vmatrix} -3 \\ 4 \end{vmatrix}} = \begin{pmatrix} -4.5 \\ 6 \end{pmatrix}.$

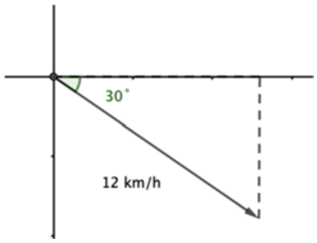
c $\dfrac{18\begin{pmatrix} -1 \\ -4 \\ 8 \end{pmatrix}}{\begin{vmatrix} -1 \\ -4 \\ 8 \end{vmatrix}} = \begin{pmatrix} -2 \\ -8 \\ 16 \end{pmatrix}.$

d $\dfrac{5\begin{pmatrix} -1 \\ -1 \end{pmatrix}}{\begin{vmatrix} -1 \\ -1 \end{vmatrix}} = \begin{pmatrix} -3.54 \\ -3.54 \end{pmatrix}.$

e Anticlockwise angle with x-axis is $90 - 40 = 50°$.

$\begin{pmatrix} 15\cos(50) \\ 15\sin(50) \end{pmatrix} = \begin{pmatrix} 9.64 \\ 11.5 \end{pmatrix}$

f Clockwise angle with x-axis is $120 - 90 = 30°$



$$\begin{pmatrix} 12\cos(30) \\ -12\sin(30) \end{pmatrix} = \begin{pmatrix} 10.4 \\ -6 \end{pmatrix}.$$

3 **a** $\sqrt{20^2+30^2} = 36.1\text{m}$

b $p = \begin{pmatrix} 20 \\ 30 \end{pmatrix} + t\begin{pmatrix} -3 \\ -5 \end{pmatrix}.$

c $d = \sqrt{(20-3t)^2+(30-5t)^2}$. From the GDC this is minimized when $t = 6.18$ seconds and the shortest distance is 1.71 m.

4 **a** $p = 3i + j + t\dfrac{10(3i-4j)}{|3i-4j|} = 3i + j + t(6i-8j).$

b $(3+6\times4)i + (1-8\times4)j = 27i - 31j.$

c $\sqrt{27^2+31^2} = 41.1\text{m}$

d $4\times10 = 40\,\text{m}$

5 **a** $(1,0,2)+3(-1,3,1) = (-2,9,5)$

b **i** $r = \begin{pmatrix} -2 \\ 9 \\ 5 \end{pmatrix} + (t-3)\begin{pmatrix} 1 \\ 4 \\ 0 \end{pmatrix}$

ii $r = \begin{pmatrix} -2 \\ 9 \\ 5 \end{pmatrix} + t'\begin{pmatrix} 1 \\ 4 \\ 0 \end{pmatrix}.$

c **i** $(-2,9,5)+2(1,4,0) = (0,17,5).$

ii $\sqrt{17^2+5^2} = 17.7$

6 **a** Compare i coefficients: $(-5+2t) = (3-2t) \Rightarrow t = 2.$ At t = 2 the j coefficients are $10+2(2) = 14$ and $4+2(5) = 14$. So the two ships collide when t = 2 (at 12:00).

b

$$\begin{aligned} a &= (-5+t)i + (10+2t)j. \\ AB = b - a &= (3-2t-t+5)i + (4+5t-10-2t)j \\ &= (8-3t)i + (3t-6)j \end{aligned}$$

c A is north of B when $(8-3t) = 0 \Rightarrow t = \dfrac{8}{3}$. So at 12:40. The distance is $3\left(\dfrac{8}{3}\right) - 6 = 2\,\text{km}$.

d $d = \sqrt{(8-3t)^2+(3t-6)^2}$. From the GDC this is minimized when $t = 2.33$. So at 12:20. At this time the distance is 1.41km.

7 a i $\dfrac{3.5\begin{pmatrix}6\\2\\3\end{pmatrix}}{\begin{vmatrix}6\\2\\3\end{vmatrix}} = \begin{pmatrix}3\\1\\1.5\end{pmatrix}$.

ii 1.5ms^{-1}

b $\begin{pmatrix}0\\0\\30\end{pmatrix} + 60\begin{pmatrix}3\\1\\1.5\end{pmatrix} = \begin{pmatrix}180\\60\\120\end{pmatrix}$.

c $r = \begin{pmatrix}180\\60\\120\end{pmatrix} + t'\begin{pmatrix}3\\1\\-0.6\end{pmatrix}$.

d $\dfrac{120}{0.6} = 200\,\text{s}$

e $\begin{vmatrix}180 + 200 \times 3\\60 + 200 \times 1\\0\end{vmatrix} = \sqrt{780^2 + 260^2} = 822\,\text{m}$.

8 a $\begin{pmatrix}0\\-9\end{pmatrix}$.

$$p_S = \begin{pmatrix}20\\15\end{pmatrix} + \begin{pmatrix}0\\-9\end{pmatrix}t$$

$$p_B = \begin{pmatrix}0\\5\end{pmatrix} + \begin{pmatrix}9\\12\end{pmatrix}t$$

b $\sqrt{(20-0)^2 + (15-5)^2} = 10\sqrt{5} = 22.4\,\text{km}$

c $BS = s - b = \begin{pmatrix}20 - 9t\\15 - 9t - 5 - 12t\end{pmatrix} = \begin{pmatrix}20 - 9t\\10 - 21t\end{pmatrix}$

d Unit vector in south-east direction is $\begin{pmatrix}1\\-1\end{pmatrix}$. BS is in the same direction as this vector, i.e.

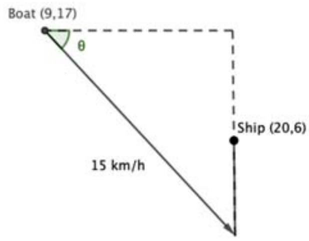
$$\tan\theta = \frac{10 - 21t}{20 - 9t} = \frac{-1}{1}$$

$-(20 - 9t) = 10 - 21t \Rightarrow t = 1$.

Boat changes direction at 11:00. Displacement is $\begin{pmatrix}9\\17\end{pmatrix}$.

e $\begin{pmatrix}20\\6\end{pmatrix}$.

f



The speed of the boat before changing direction was $\sqrt{9^2+12^2}=15$

If the direction of the boat after the change is at an angle of θ, clockwise from east then its position is given by $r=\begin{pmatrix}9\\17\end{pmatrix}+15t'\begin{pmatrix}\cos(\theta)\\-\sin(\theta)\end{pmatrix}$ where t' is the time after it changes direction. The ship's position is given by $s=\begin{pmatrix}20\\6\end{pmatrix}+\begin{pmatrix}0\\-9\end{pmatrix}t'$.

When the boat intercepts the ship

$20=9+15t'\cos(\theta)\Rightarrow t'=\dfrac{11}{15\cos(\theta)}$.

$6-9t'=17-15t'\sin(\theta)\Rightarrow 15t'\sin(\theta)=11+9t'$

.

Since $t'=\dfrac{11}{15\cos(\theta)}$,we can substitute to get

$\dfrac{11\sin(\theta)}{\cos(\theta)}=11+\dfrac{99}{15\cos(\theta)}\Rightarrow 11\sin(\theta)-11\cos(\theta)=\dfrac{99}{15}\Rightarrow \sin(\theta)-\cos(\theta)=\dfrac{3}{5}$

Solve on a GDC to obtain, $\theta=70.1^o$, hence bearing is 160.1°.

Chapter review

1 **a** $\sqrt{(1.2+0.2)^2+(8.5-9.4)^2+(3.1-2.6)^2}=1.74\,\text{km}$

b First aircraft: $\sqrt{1.2^2+8.5^2+3.1^2}=9.13\,\text{km}$

Second aircraft: $\sqrt{0.2^2+9.4^2+2.6^2}=9.75\,\text{km}$. Second aircraft is further.

2 **a** $m=\dfrac{3-5}{6-1}=-\dfrac{2}{5}=-0.4.\quad y=-0.4x+c.\quad 5=-0.4+c\Rightarrow c=5.4\quad y=-0.4x+5.4$

b $5x-2(-0.4x+5.4)+5=0\Rightarrow 5.8x=5.8\Rightarrow x=1,\ y=5.\quad (1,5)$.

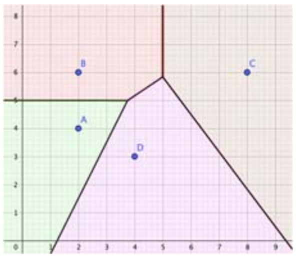
3 **a** Perpendicular bisector of [AB] is y = 4 and of [BC] is x = 5.

b Midpoint of [AC] is $\left(\frac{2+8}{2}, \frac{4+6}{2}\right) = (5,5)$. $m = -\frac{8-2}{6-4} = -3$. $y = -3x + 20$

c

d Midpoint of [AD] is $\left(\frac{2+4}{2}, \frac{4+3}{2}\right) = (3, 3.5)$, $m = -\frac{4-2}{3-4} = 2$. $y = 2x - 2.5$.

e



4 a Perpendicular bisector of [BC]: Midpoint $\left(\frac{6+3}{2}, \frac{5+2}{2}\right) = (4.5, 3.5)$. $m = -\frac{6-3}{5-2} = -1$.

$y = -x + 8$.

Perpendicular bisector of [AC]: Midpoint $\left(\frac{2+3}{2}, \frac{4+2}{2}\right) = (2.5, 3)$. $m = -\frac{2-3}{4-2} = \frac{1}{2}$.

$y = \frac{1}{2}x + \frac{7}{4}$.

b Should be placed at the intersection of the perpendicular bisectors: $\frac{1}{2}x + \frac{7}{4} = -x + 8$

$\Rightarrow \frac{3}{2}x = \frac{25}{4} \Rightarrow x = 4.17,\ y = 3.83$. Should be built at (4.17,3.83).

c Distance is the same to each of the previous three outlets and is

$\sqrt{(2-4.17)^2 + (4-3.83)^2} = 2.18$ km

5 a t = 0: $\begin{pmatrix} 5 \\ 6 \end{pmatrix}$.

b t = 2: $\begin{pmatrix} -3 \\ 0 \end{pmatrix}$.

c $\sqrt{4^2 + 3^2} = 5$ m min^{-1}.

d $\sqrt{(5-4t)^2 + (6-3t)^2}$. From the GDC the minimum of this is when $t = 1.52$ min. The distance is 1.8 m.

6 a

$$\begin{aligned} a\cdot b &= 1\times 2+2\times(q-1)+p\times 1\\ &= 2+2q-2+p\\ &= 2q+p \end{aligned}$$

$2q+p=0.$

b

$b=ka \Rightarrow 2=k\times 1 \Rightarrow k=2$
$q-1=2\times 2 \Rightarrow q=5$
$1=2\times p \Rightarrow p=0.5$

c

$$a=\begin{pmatrix}1\\2\\3\end{pmatrix},\ b=\begin{pmatrix}2\\1\\1\end{pmatrix}$$

$$\frac{a\cdot b}{|a||b|}=\frac{(1\times 2)+(2\times 1)+(3\times 1)}{\sqrt{(1^2+2^2+3^2)(2^2+1^2+1^2)}}=\frac{7}{\sqrt{84}}.\qquad \theta=\cos^{-1}\left(\frac{7}{\sqrt{84}}\right)=40.2°.$$

7 Two sides of the base triangle are given by the vectors $BA=a-b=\begin{pmatrix}2.5\\-2.0\\0\end{pmatrix}$ and

$CA=a-c=\begin{pmatrix}1.0\\-1.5\\0\end{pmatrix}$ so the area of the triangular base is

$\frac{1}{2}\left|\begin{pmatrix}2.5\\-2\\0\end{pmatrix}\times\begin{pmatrix}1\\-1.5\\0\end{pmatrix}\right|=\frac{1}{2}\left|\begin{matrix}-2\times 0-0\times -1.5\\1\times 0-2.5\times 0\\2.5\times -1.5-1\times -2\end{matrix}\right|=\frac{1}{2}\left|\begin{matrix}0\\0\\-1.75\end{matrix}\right|=0.875$. The height of the tetrahedron is 2 so

the volume is $\frac{1}{3}\times 2\times 0.875=0.583.$

8 a $-8=3-t\Rightarrow t=11.\qquad n=2+2t=2+22=24.$

b $1-s=-8\Rightarrow s=9.$

Hence, $\begin{pmatrix}1\\-2\\u\end{pmatrix}+9\begin{pmatrix}-1\\p\\q\end{pmatrix}=\begin{pmatrix}-8\\34\\24\end{pmatrix}$

$-2+9p=34\Rightarrow p=4.$

$\begin{pmatrix}-1\\4\\q\end{pmatrix}\cdot\begin{pmatrix}-1\\3\\2\end{pmatrix}=1+12+2q=0\Rightarrow q=-6.5$

$u + 9q = 24$
$u = 24 - 9 \times (-6.5) = 24 + 58.5 = 82.5.$

$p = 4,\ q = -6.5,\ u = 82.5$

9 **a** $r_A = \begin{pmatrix} 0 \\ 0 \\ 8.2 \end{pmatrix} + t\begin{pmatrix} 750\cos 45 \\ 750\sin 45 \\ 0 \end{pmatrix} + t\begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 8.2 \end{pmatrix} + t\begin{pmatrix} 530.3 \\ 530.3 \\ 2 \end{pmatrix}.$

b $r_B = \begin{pmatrix} 0 \\ 0 \\ 13.1 \end{pmatrix} + (t - 0.5)\begin{pmatrix} 800\sin(30) \\ 800\cos(30) \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 13.1 \end{pmatrix} + (t - 0.5)\begin{pmatrix} 400 \\ 693 \\ -1 \end{pmatrix}.$

c $8.2 + 2t = 13.1 - (t - 0.5) \Rightarrow 3t = 5.4 \Rightarrow t = 1.8.$

$r_A = \begin{pmatrix} 0 \\ 0 \\ 8.2 \end{pmatrix} + 1.8 \times \begin{pmatrix} 530.3 \\ 530.3 \\ 2 \end{pmatrix}.$

$r_B = \begin{pmatrix} 0 \\ 0 \\ 13.1 \end{pmatrix} + 1.3 \times \begin{pmatrix} 400 \\ 693 \\ -1 \end{pmatrix}.$

Ignoring the height,

$d = \sqrt{(1.8 \times 530.3 - 1.3 \times 400)^2 + (1.8 \times 530.3 - 1.3 \times 693)^2} = 438\text{km}.$

Exam style Questions

10 a M is $\left(\frac{-3+5}{2}, \frac{8+3}{2}\right) = (1, 5.5).$

b $m = \frac{3-8}{5--3} = -\frac{5}{8} = -0.625.$

c **i** $\frac{8}{5} = 1.6.$

ii $y = 1.6x + c.$ $5.5 = 1.6 + c \Rightarrow c = 3.9.$ $y = 1.6x + 3.9.$

11 i Neither, as the gradients are not the same and their product is not −1.

ii Parallel, as both lines have a gradient of 3.

iii Neither, as the gradients are not the same and their product is not −1.

iv Perpendicular, as the product of the gradients is −1.

v Perpendicular, as the product of the gradients is −1.

12 a $\sqrt{500^2 + 400^2 + 300^2} = 707\text{m}$

b $707.10\ldots + \sqrt{400^2 + 200^2 + 400^2} = 1310\text{m}$

13 a $AB = b - a = \begin{pmatrix}1\\1\\0\end{pmatrix}$, $\quad AC = c - a = \begin{pmatrix}3\\3\\1\end{pmatrix}$. $\quad AB \times AC = \begin{pmatrix}1\times1-0\times3\\0\times3-1\times1\\1\times3-1\times3\end{pmatrix} = \begin{pmatrix}1\\-1\\0\end{pmatrix}$.

b The area is $\frac{1}{2}\times\left|AB\times AC\right| = \frac{1}{2}\times\left|\begin{matrix}1\\-1\\0\end{matrix}\right| = \frac{1}{2}\times\sqrt{2} = \frac{\sqrt{2}}{2}$.

14 a $\frac{a}{3}\times\frac{2}{3} = -1 \Rightarrow a = -4.5$.

b

$$\begin{aligned}-4.5x - 3\left(\frac{2}{3}x+4\right) &= 9\\ -6.5x - 12 &= 9\\ -6.5x &= 21\\ x &= -3.23\\ y &= \frac{2}{3}x + 4 = 1.85.\end{aligned}$$

$(-3.23, 1.85)$.

15 a $AB = 5$, $\quad AF = 6$, $\quad AO = 3$. $\quad$ Surface area is $2\times(5\times6+5\times3+3\times6) = 126$.

b $\sqrt{3^2+5^2+6^2} = \sqrt{70} = 8.37$.

c i $\left(\frac{3+0}{2}, \frac{5+0}{2}, \frac{6+0}{2}\right) = (1.5, 2.5, 3)$.

ii $x = AM = m - a = \begin{pmatrix}-1.5\\2.5\\3\end{pmatrix}$, $\quad y = BM = m - b = \begin{pmatrix}-1.5\\-2.5\\3\end{pmatrix}$.

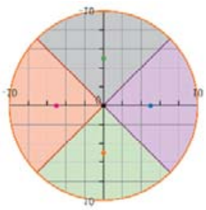
$$\frac{x\cdot y}{|x||y|} = \frac{-1.5\times-1.5+2.5\times-2.5+3\times3}{\sqrt{(1.5^2+2.5^2+3^2)(1.5^2+2.5^2+3^2)}} = \frac{5}{17.5}.$$

$$\theta = \cos^{-1}\left(\frac{5}{17.5}\right) = 73.4^\circ.$$

16 a The distance to the town from a point (x,y) on the road is given by $d = \sqrt{(80-x)^2+(140-y)^2} = \sqrt{(80-x)^2+(140-x+80)^2} = \sqrt{2x^2-600x+54800}$. This is minimized when $x = \frac{600}{2\times2} = 150$. $\quad y = 150 - 80 = 70$. The new airport is at (150,70).

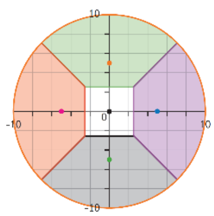
b The distance between the town and airport is $\sqrt{(80-150)^2+(140-70)^2} = 99.0\text{km}$

17 a



b $\frac{1}{4} \times \pi \times 10^2 = 78.5\,\text{m}^2$

c



d $5 \times 5 = 25\,\text{m}^2$.

e $\frac{1}{4} \times \left(\pi \times 10^2 - 25\right) = \frac{1}{4} \times 289.16 = 72.3\,\text{m}^2$.

f 4.

18 a $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + 4\begin{pmatrix} 50 \\ 60 \\ 1 \end{pmatrix} = \begin{pmatrix} 200 \\ 240 \\ 5 \end{pmatrix}$.

b $1 + t = \lambda \Rightarrow s = \begin{pmatrix} -90 \\ -100 \\ 0 \end{pmatrix} + (1+t)\begin{pmatrix} 60 \\ 70 \\ 1 \end{pmatrix} = \begin{pmatrix} -30 \\ -30 \\ 1 \end{pmatrix} + t\begin{pmatrix} 60 \\ 70 \\ 1 \end{pmatrix}$. $50t = -30 + 60t \Rightarrow 10t = 30, \Rightarrow t = 3$.

When t is 3 the first aircraft is at $\begin{pmatrix} 150 \\ 180 \\ 4 \end{pmatrix}$, and when λ is 4 the second aircraft is at $\begin{pmatrix} 150 \\ 180 \\ 4 \end{pmatrix}$,

so the two flightpaths do cross and cross at $\begin{pmatrix} 150 \\ 180 \\ 4 \end{pmatrix}$.

c The two aircrafts do not collide as the first aircraft gets to the intersection point an hour before the second does.

d $\sqrt{90^2 + 100^2 + 1^2} = \sqrt{18101} = 135\text{km}$.

e The distance between the aircraft is given by

$$d = \sqrt{(50t + 90 - 60t)^2 + (60t + 100 - 70t) + (1 + t - t)^2} = \sqrt{(90 - 10t)^2 + (100 - 10t)^2 + 1^2}$$

This is minimized when $t = 9.5$ and at this time the distance is 7.14km. So the shortest distance between the planes is 7.14 km after 9 and a half hours.

Paper 3

a	**i**	Coordinates of B are $\left(6370\cos 50^\circ, 6370\sin 50^\circ\right)$	M1A1
		$4090, 4880$	
	ii	$\frac{50}{360}\times 2\pi\times 6370 = 5560$	M1A1
	iii	$\sqrt{\left(6370-4090\right)^2+4880^2}$	M1A1
		= 5380 km	A1
	iv	Percentage error = $\frac{5560-5380}{5560}\times 100 = 3.2\%$	M1A1

[9 marks]

b	**i**	$R\sin\theta$	A1
	ii	Projection of *P* onto the plane of the equator is a distance $R\cos\theta$ from O	(M1)
		Coordinates are $\left(R\cos\theta\cos\phi, R\cos\theta\sin\phi, R\sin\theta\right)$	A1A1

[4 marks]

c **i**

$$\left(6370\cos\left(-12.05\right)\cos\left(-77.04\right), 6370\cos\left(-12.05\right)\sin\left(-77.04\right), 6370\sin\left(-12.05\right)\right)$$

M1A1

$=\left(1397, -6071, -1329\right)$ AG

[2 marks]

d **i** 6370 km A1

ii

$$\begin{array}{cc} 1397 & 3945 \\ 6071 & 3346 \\ 1329 & 3717 \end{array} \quad 30764624$$

M1(A1)

Using the fact that both cities lie on the surface of the earth, and hence have magnitude 6370, gives

$$\cos\left(\mathrm{L\hat{O}P}\right) = \frac{-30764624}{6370^2}\left(= -0.758\right)$$

M1A1

= 139.3˚ A1

Note: It is also possible to solve **ii** by finding the straight-line distance between Lima and Tokyo, and use the cosine rule or right-angled trigonometry.

iii

$$\frac{139.3}{360} \times 2\pi \times 6370$$

(M1)

= 15 490 km A1

[8 marks]

e **i** Distances form arcs of a circle subtending the same angle and are hence proportional to the radius. (M1)

Upper bound $15\,487 \quad \frac{6384}{6370} \quad 15\,520$ km A1

Lower bound $15\,487 \quad \frac{6353}{6370} \quad 15\,450$ km A1

ii Both are approximately 15 500 km to 3 significant figures. A1

[4 marks]

Total: 27 marks

4 Modelling constant rates of change: linear functions and regressions

Skills check

1 a $y = 0.25x + 1.75$ **b** $y = -3x - 4$

2 a $y = 23$ **b** $x = 4$

Exercise 4A

1 a Not a function, because one x-value may have multiple y-values. As there are 500 people and 12 months, at least 1 month will have two or more people with a birthday in it.

b y is a function of x, because for each month, there is exactly one value for the number of people with a birthday in it. For example, if 42 people have birthdays in January, then x = 1 corresponds to y = 42.

2 a R_3 is a function. Every element from the first set, A, is mapped onto only one element from the second set, B.

b Not a function. In R_4, the input 3 has two outputs, 1 and 2.

c Not a function as it fails the vertical line test

d Is a function, passes the vertical line test

e Is a function, for each x value, there is a unique y value

f Not a function as it fails the vertical line test.

3 a $R : x \to \frac{1}{x}$

b Yes.

c $\frac{1}{2}$

d $a = \frac{1}{3}$

4 a $(-1)^3 = -1$

b $x^3 = -64$
$x = \sqrt[3]{-64} \Rightarrow x = -4$

c $B = \{-1, 0, 1, 8, 27, 64\}$

d It is a function, because for each x value, there is a unique y value.

Exercise 4B

1 a i 5 **ii** -3

b i 0 **ii** 3

c $x > 3$

2 **a** **i** $f(2) = 10 - 4(2) = 2$

ii $f\left(-\frac{1}{2}\right) = 10 - 4\left(-\frac{1}{2}\right) = 12$

b $f(2.5) = 10 - 4(2.5) = 0$

c $10 - 4x = -6$
$-4x = -16$
$x = 4$

3 **a** **i** $F(0) \approx 687$ N . This is Jaime's weight at sea level.

ii $F(410) \approx 607$ N

iii $\frac{607}{687} \times 100 = 88.4\%$
$100\% - 88.4\% = 11.6\%$

Jaime is 11.6% lighter on the space station.

b When F = 625, h ≈ 310 km. The force of gravity on Jaime is 625 N at a height of 310 km above sea level.

c $0.95 \times 687 = 653$

F = 653 when $h \approx 170\,\text{km}$ above sea level.

$F(653) = 170$

Exercise 4C

1 **a** On January 2nd last year the average temperature was 25°C.

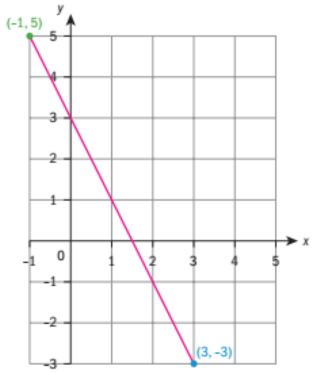
b Domain = {1, 2, 3, …, 31}

c Answers need to include all the values in the table so should be similar to $20 \le T \le 30$, or $T \in [20, 30]$ if it can take any value

2 **a** **i** $f(-1) = -2(-1) + 3 = 5$ (ii) $f(3) = -2(3) + 3 = -3$

b $-2x + 3 = 2$
$-2x = -1$
$x = \frac{1}{2}$

c



d $-3 \le y \le 5$ or $y \in [-3, 5]$

3 **a** Domain = {−5, −4, −3, −2, −1, 0, 1, 2, 4}

Range = {−2, 0, 2, 4, 6, 8}

b Domain : $-8 \le x \le 6$

Range: $-4 \le y \le 3$

c Domain: $-7 < x \le 9$

Range: $0 \le y < 4$

d Domain: $-7 < x < 7$

Range: $-4 \le y < 3$

4 **a** $p \in \{3, 4, 5, 6, 7, 8\}$

b $w(3) = 9 + \frac{15}{3+1} = 12.8\text{kg}$, $w(8) = 9 + \frac{15}{8+1} = 10.7\text{kg}$. If there are 3 people in the group, each must carry 12.8 kg of food. If there are 8 people in the group, each must carry 10.7 kg of food.

c The function goes from (3, 12.8) to (8, 10.7), so the range is $\{w \mid 10.7 \le w \le 12.8\}$

d

$$11 = 9 + \frac{15}{p+1}$$

$$p = \frac{13}{2} = 6.5$$

So, $p \le 6$.

$$12 = 9 + \frac{15}{p+1}$$

$$p = 4$$

Hence, $p \in \{4, 5, 6\}$. So, Robbin can take from 4 to 6 people.

5 **a** $0 \le t \le 5$ as valid up to January 2019

b $S(0) = \$3.81$ (given)

$$S(5) = -0.09(5)^2 - 0.0651(5) + 3.81$$
$$= \$1.23$$

$$1.23 \le S \le 3.81$$

c $S(2.5) = -0.09(2.5)^2 - 0.0651(2.5) + 3.81 = \$3.08$

d $-0.09t^2 - 0.0651t + 3.81 = 1.5$
$-0.09t^2 - 0.0651t + 2.31 = 0$
$t = 4.72$

During 2018

e The researchers would be extrapolating far beyond the data set; also the values of the function become negative after 6 years.

6 a $R(2500) = \frac{1.75}{10000}(2500)(30000 - 2500) \approx$ €12031

b $R(5000) = \frac{1.75}{10000}(5000)(30000 - 5000) \approx 21875$

$21875 - 12031 =$ €9844

c $27000 - \frac{60000000}{q + 5000} = 20000$

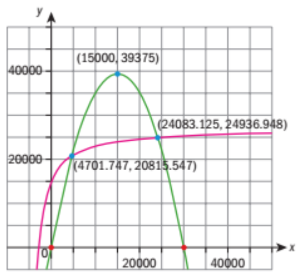
$\frac{60000000}{q + 5000} = 7000$

$q + 5000 = \frac{60000000}{7000}$

$q = 3571$

The cost to produce 3571 bottles is €20000.

d



One break-even point is at q = 4702 and the other is at q = 24083

e Company makes a profit when $\{q \mid 4702 \le q \le 24083\}$

7 a $V(0) = 8500 + 5(-15)^2 = 9625$, a = £9625

$V(1) = \frac{8500}{2} + 5(-13)^2 = 5230$, b = £5230

b Technology shows that the value drops below after 9.2 years so during 2017

c She bought it for £9625. According to technology, it will be worth that again after 58.5 years, so during 2066.

Exericse 4D

1 a linear with gradient zero

b linear with gradient −2

c not linear

d linear with gradient 5

2 a i independent = time parked (in hours)

dependent = cost of parking (in $)

ii Linear, with a constant rate of change of $12.50 per hour

b **i** independent = time (in years)

dependent = population (in number of fish)

ii Not linear, as a decline of 7% of the current population will not be a constant number of fish per year

c **i** independent = amount of purchase (euros)

dependent = amount of tax (euros)

ii Linear, with a constant rate of change of €0.22 tax per euro spent

d **i** independent = daily high temperature (in $°C$)

dependent = number of daily passes sold (number of passes)

ii The relationship is linear, because the rate of change between any two points of the function is constant and equivalent to a decrease of 8 passes per degree Celsius.

3 It is not a linear function. The rate of change from 1 to 3 is $\frac{4}{2} = 2$. The rate of change from 5 to 8 is $\frac{4}{3} \neq 2$

4 **a** US$30

b $P(1.5) = 100(1.5) + 30 =$ US$180

c $100x + 30 = 310$
$100x = 280$
$x = 2.8$ kg

5 **a** $d(t) = 13 - 0.065t$

b $d(60) = 13 - 0.065(60) = 9.1$ km

c $13 - 0.065t = 0$
$0.065t = 13$
$t = 200$

Ewout will take 200 minutes or approximately 3 and a half hours to reach home.

6 **a** $64\% - 62.65\% = 1.35\%$

$S(t) = 64 - 1.35t$

b $64 - 1.35t = 50$
$1.35t = 14$
$t = 10.37$

10.37 years after 2018, so during 2028

c $\frac{18-13}{2-0} = 2.5$

Foot Talker grows at 2.5% of total sales per year, whereas Sneakies declines by 1.35% of total sales.

d $F(t) = 13 + 2.5t$

e $13 + 2.5t = 64 - 1.35t$
$3.85t = 51$
$t = 13.2$

13.2 years after 2018, so during 2031

7 **a** 120 m s^{-1} **b** 8 sec **c** $\frac{-120}{8} = -15\text{ m s}^{-2}$

d $v(t) = -15t + 120$

8 **a** $50m + c = 20$ **b** $80m + c = 35$

c $30m = 15$
$m = 0.5$
$50(0.5) + c = 20$
$c = 20 - 25 = -5$
$L = 0.5W - 5$

d $L = 0.5(90) - 5 = 40\,\text{cm}$

Exercise 4E

1 **a** $c = \frac{1}{0.21}a$ or $c = 4.76a$

b one AUD is worth 4.76 CNY

c $599 = 4.76a$
$a = 125.84$
$126 - 75 = 51$

51 AUD needed

2 **a** $15 = 0.64k$
$k = 23.4$
$F(x) = 23.4x$

b $80 = 23.4x \quad x = 3.42\,\text{m}$

c $23.4(-1.5) = -35.1\,N$

Exercise 4F

1 **a** $u(x) = 1.33x$

b $1.33x = 100$
$x = £75.19$

c **i** $u(500) = 665$, $665 - 661.72 = \$3.28$

ii $B(x) = 1.33x - 3.28$

iii $1.33x - 3.28 = 1000$
$1.33x = 1003.28$
$x = £754.35$

2 **a** gradient: for each 1 euro that the price increases, the number of people willing to buy tickets deceases by 36.

y-intercept: 5000 people will buy the tickets if they cost 0 euros.

b $N(75) = 5000 - 36(75) = 2300$

c $5000 - 36p = 0$
$36p = 5000$
$p = 138.89$ euros

This is the price at which no one is willing to buy a ticket.

d Domain $\{p \mid 0 \le p \le 139\}$, range $\{N \mid 0 \le N \le 5000\}$

e $5000 - 36p = 28p - 504$
$5504 = 64p$
$p = 86$ euros

3 a i $P(0.918) = -0.107(0.918) + 1 = 0.902$, 90.2% of sea level

ii $B(0.918) = -11.7(0.918) + 100 = 89.3°C$

b There is a constant rate of change between the two variables, cooking time and boiling point temperature

c $T(B) = 15 + \frac{2}{5}(100 - B)$

d $a = 5.130,\ B(5.130) = -11.7(5.130) + 100 = 39.979°C$

$T(39.979) = 15 + \frac{2}{5}(100 - 39.979) = 39$ minutes

24 minutes longer

e Domain $\{B \mid 40 \le B \le 100\}$, Range $\{T \mid 15 \le T \le 39\}$

4 a x = number of portions of pepperoni, where 1 portion is 100 g

y = number of portions of Parma ham, where 1 portion is 100 g

$3.5x + 6.5y = 25$

b $y = 0 \to x = \frac{25}{3.5} = 7.14$. Alfie can buy 714 g of just pepperoni

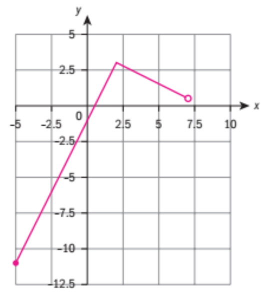
$x = 0 \to y = \frac{25}{6.5} = 3.85$. Alfie can buy 385 g of just Parma ham

c $x = y + 2$
$3.5(y + 2) + 6.5y = 25$
$10y + 7 = 25$
$y = 1.8$
$x = 3.8$

Alfie should buy 180 g of Parma ham and 380 g of pepperoni

Exercise 4G

1 a



b **i** $f(5.7) = 4 - \frac{1}{2}(5.7) = 1.15$, (ii) $f(-3.2) = 2(-3.2) - 1 = -7.4$

c $2x - 1 = 2$
$2x = 3$
$x = 1.5$

$4 - \frac{1}{2}x = 2$
$\frac{1}{2}x = 2$
$x = 4$

d The pieces connect at x = 2; $2(2) - 1 = 3$ and $4 - \frac{1}{2}(2) = 3$. Outputs match.

e Domain $\{x \mid -5 \le x < 7\}$, Range $\{y \mid -11 \le y \le 3\}$ since the highest point is at (2,3)

2 **a** $f(x) = \begin{cases} 130x & 0 \le x < 5 \\ 650 & 5 \le x < 7 \\ 130x - 260 & 7 \le x \le 10 \end{cases}$

b $f(5) = 130(5) = 650$
$f(7) = 130(7) - 260 = 650$

The pieces connect so f is continuous.

c $f(x) = 300$
$130x = 300$
$x = 2.31$

$f(x) = 800$
$130x - 260 = 800$
$130x = 1060$
$x = 8.15$

It took Amir $8.15 - 2.31 = 5.84$ minutes

3 **a** The additional cost is 6 cents per megabyte or \$0.06 × 1,000 per gigabyte = \$60 per gigabyte.

$C(d) = \begin{cases} 35 & 0 \le d \le 1 \\ 35 + 60(d - 1) & d > 1 \end{cases}$

b **i** $C(0.5) = \$35$

ii $C(2) = 35 + 60(2 - 1) = \$95$

c $C(d) = 59$
$35 + 60(d - 1) = 59$
$60d - 60 = 24$
$60d = 84$
$d = 1.40$

The unlimited plan is better if more than 1.4 GB of data are used.

d i $\frac{172}{3} \times 31 \approx 1777$ or approximately 1.78 GB

ii The plan charging \$59 per month is cheaper, because 1.78 > 1.4.

$C(1.78) = 35 + 60(1.78 - 1) = 81.8$. She would save \$22.80 $\approx \$23$.

Exercise 4H

1 a $f(x) = 4$
$3x + 5 = 4$
$3x = -1$
$x = -\frac{1}{3}$

b $f(x) = 4$
$\frac{1}{2}x = 4$
$x = 8$

c Read from graph, $x \approx 2$

d $x = -6$

2 a Domain: $-6 \le x \le 6$, Range: $2 \le y \le 8$

b Read from graph, $g(4) \approx -1$

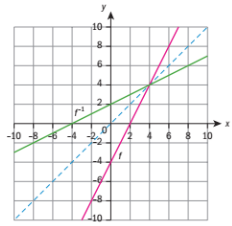
3 $C^{-1}(8) \to C(r) = 8$
$2\pi r = 8$
$r = \frac{8}{2\pi} = 1.27$

A circle with circumference 8 units will have a radius of 1.27 units.

4 $f^{-1}(21) = -5$
$f(-5) = 21$
$2(-5) + c = 21$
$c = 31$

Exercise 4I

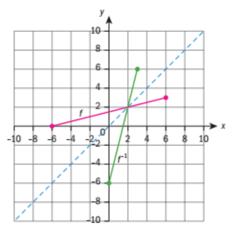
1 a i



ii Domain: $x \in$, Range: $y \in$

iii $x = 4$

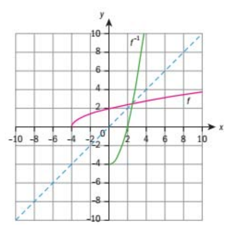
b **i**



ii Domain: $\{x \mid 0 \le x \le 3\}$, Range $\{y \mid -6 \le y \le 6\}$

iii $x = 2$

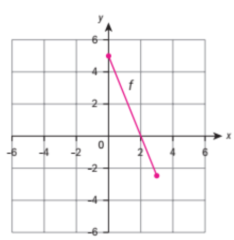
c **i**



ii Domain: $\{x \mid x \ge 0\}$, Range: $\{y \mid y \ge -4\}$

iii $x = 2.5$

2 **a**



b $f^{-1}(b) = 3$

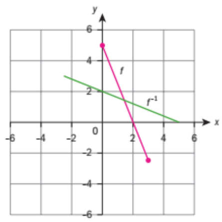
$f(3) = -2.5(3) + 5 = -2.5$

c

Function	Domain	Range

f	$0 \le x \le 3$	$-2.5 \le y \le 5$
f^{-1}	$-2.5 \le x \le 5$	$0 \le y \le 3$

d



e $f(x) = f^{-1}(x) = x$

$-2.5x + 5 = x$

$5 = 3.5x$

$x = \frac{10}{7}$

$\left(\frac{10}{7}, \frac{10}{7}\right)$

3 a $d(x) = 90x + 630$

b $90x + 630 = y$

$90x = y - 630$

$x = \frac{y - 630}{90} = \frac{y}{90} - 7$

$d^{-1}(x) = \frac{x}{90} - 7$

c i $d^{-1}(700) = \frac{700}{90} - 7 = 0.78 \approx 47\text{min}$ at 8:47AM

ii $d^{-1}(880) = \frac{880}{90} - 7 = 2.78$ at 10:47AM

iii $d^{-1}(1400) = \frac{1400}{90} - 7 = 8.56 =$ 8 hours 34 min; at 4:34PM

d Prague **ii** is most reasonable as it is closest to the middle of the day.

Exercise 4J

1 a $f \quad g(1) = f(4 - 1)$

$= 8(3) - 25$

$= -1$

b $h \quad f(4) = h(8(4) - 25)$

$= h(7)$

$= \frac{3}{2}(7)$

$= 10.5$

c $g \quad f(x) = g(8x - 25)$

$= 4 - (8x - 25)$

$= 29 - 8x$

d $h \quad h \quad h(x)=\frac{3}{2}\left(\frac{3}{2}\left(\frac{3}{2}x\right)\right)=\frac{27}{8}x$

2 **a** $c(t(x))=1.1x+2$
$t(c(x))=1.1(x+2)=1.1x+2.2$

These are not equal because they are parallel linear functions with different y-intercepts.

b $t(c(x))$ provides a 0.20 euro larger tip, because 2.20 – 2.00 = 0.20 and the other parts of the two functions are equal.

3 Answers can vary; samples below

a $g(x)=\frac{7}{3}x$ and $f(x)=x-5$

b $g(x)=x-2$ and $f(x)=4x$

c $h(x)\approx 12-4x$, so one possibility is $g(x)=4x$ and $f(x)=12-x$

4 **a** $P(w(4))=P(7(4)-3)$
$=P(25)$
$=50(25)-2000$
$=-750$

the company loses $750 if it operates for exactly 4 hours

b $P(w(t))=P(7t-3)$
$=50(7t-3)-2000$
$=350t-150-2000$
$=350t-2150$

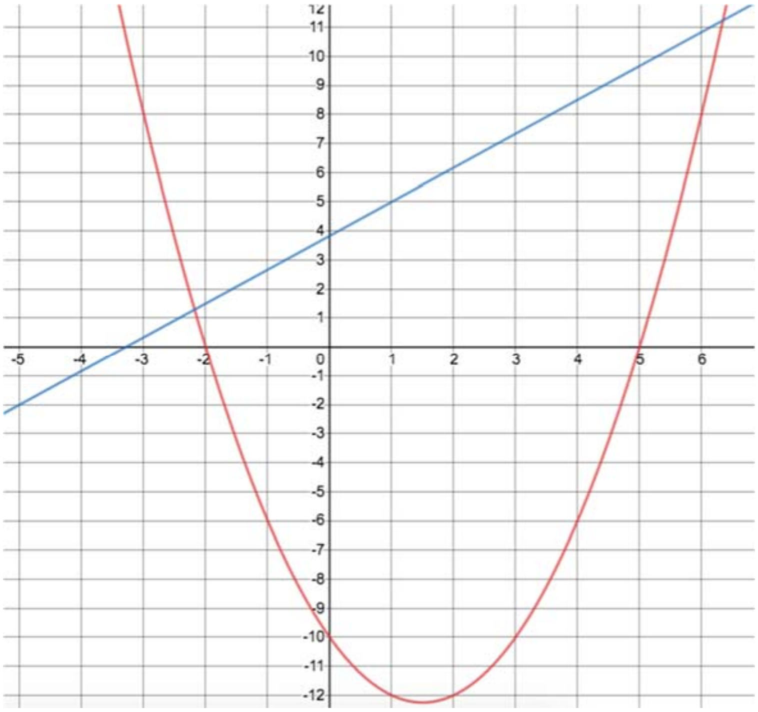
c $P(t)>0$
$350t-2150>0$
$350t>2150$
$t>6.14$

The company must operate for at least 7 hours each day.

5 $C \quad F(x)=C\left(\frac{9}{5}x+32\right)$
$=\frac{5}{9}\left(\frac{9}{5}x+32\right)-17.8$
$=x+17.8-17.8$
$=x$
$=i(x)$

Similarly, $F \quad C(x)=x$

6 Reading from graph



$f(x) = 0$
$x = -2, 5$

$g(x) = -2$
$x = -5$
$g(x) = 5$
$x = 1$

So $(f \circ g)(x) = 0$ when $x = -5$ and $x = 1$

Exercise 4K

1 **a** **i** 41.7, 50.3, 58.9 **ii** yes, d = 8.6

b **i** −83, −105, −127 **ii** yes, d = −22

c **i** 151, 196, 250

ii no, the difference is not constant, it increases by 9 each time

d **i** 1.25, 0.625, 0.3125

ii no, the numbers are divided by 2 each time instead of being added or subtracted by a constant amount.

2 **a** 9.5, 12, 14.5. This is arithmetic; $a_1 = 9.5$, $d = 2.5$

b 2650, 300, −2050. This is arithmetic; $b_1 = 2650$, $d = -2350$

c 3, 10, 21. Not arithmetic, no common difference

3 **a** $u_n = 5 + 4(n-1)$, $u_n = 4n + 1$

b $116 = 4n + 1$ has solution $n = 28.75$. This is not a whole number, so 116 is not a term of the sequence.

4 **a** 95 in the second year, 105 in the third

b $u_{10} = 85 + 10(10-1) = 175$ employees

c $u_n = 75 + 10n$

$75 + 10n = 285$

$10n = 210$

$n = 21$

21 years after opening.

5 **a** $a_n = 2.6 + 1.22(n-1)$

b $a_0 \to 1997$

$a_{28} \to 2025$

$a_{28} = 2.6 + 1.22(28-1)$

$= 35.54$ m

c $2.6 + 1.22(n-1) \geq 84$

$1.22n - 1.22 \geq 81.4$

$1.22n \geq 82.62$

$n \geq 67.72$

$n = 68$

In 2065.

Exercise 4L

1 **a** $u_1 = -10$

$u_7 = u_1 + 6d = 1$

$-10 + 6d = 1$

$6d = 11$

$d = \frac{11}{6}$

b $u_{15} = u_1 + 14d = -10 + 14\left(\frac{11}{6}\right) = \frac{47}{3}$

$u_5 = u_1 + 4d$

2 **a** $0 + 4d = 10$

$d = 2.5$

b $u_3 = u_1 + 2d = 0 + 2(2.5) = 5$

3 **a** $d = -25$. The frog hops 25 cm closer to the finish line with each hop.

b $u_{10} = u_1 + 9d = 975 + 9(-25) = 750$. After 10 hops, the frog is 750 cm from the finish line.

c $u_n = 1000 - 25n$

$1000 - 25n = 0$

$25n = 1000$

$n = 40$

It takes 40 hops to finish the race.

d $1000 - 25n = -225$
$25n = 1225$
$n = 49$

The frog hops 49 times.

4 **a** $12 = u_1 + 2d$
$43.5 = u_1 + 9d$

b $31.5 = 7d$
$d = 4.5$

$12 = u_1 + 2d$
$12 = u_1 + 9$
$u_1 = 3$

c $u_{100} = u_1 + 99d = 3 + 99(4.5) = 448.5$

5 **a** $u_1 = 22$
$u_{10} = 22 + 9d = 49$
$9d = 27$
$d = 3$
$u_n = 22 + 3(n-1) = 3n + 19$

b $3n + 19 = 106$
$3n = 87$
$n = 29$

There are 29 rows in the theatre.

6 **a** $u_1 = 12$ m

b $u_3 = 12 + 2d$
$u_9 = 12 + 8d$

c $u_9 = 2u_3$

d $12 + 8d = 2(12 + 2d)$
$12 + 8d = 24 + 4d$
$4d = 12$
$d = 3$

e $u_n = 12 + 3(n-1) = 3n + 9$
$3n + 9 = 100$
$3n = 91$
$n = 30.3$

So, Tyler should use an object that is at least 31 kg.

Exercise 4M

1 $I = 9000 \times 0.059 \times 3$
$= \$1593$

Total = 9000 + 1593 = $10593

2 $9000 = P \times 0.075 \times 7$
$P = £17142.86$

3 $1840 = 8000 \times r \times 5$
$r = 0.046 = 4.6\%$

4 To double money need \$8600 interest

$8600 = 8600 \times 0.065 \times n$
$n = 15.4$

16th year

Exercise 4N

1 $S_n = \left(2u_1 + (n-1)d\right) \times \frac{n}{2}$

$$S_{20} = \left(2(6) + (20-1)(-3)\right) \times \frac{20}{2}$$
$$= -450$$

2 $S_n = \left(2u_1 + (n-1)d\right) \times \frac{n}{2}$

$$S_{30} = \left(2(8) + (30-1)(8)\right) \times \frac{30}{2}$$
$$= 3720$$

3 **a** $u_n = 52 + 10(n-1) = 10n + 42$
$10n + 42 = 462$
$10n = 420$
$n = 42$

b $S_n = \left(2u_1 + (n-1)d\right) \times \frac{n}{2}$

$$S_{42} = \left(2(52) + (42-1)(10)\right) \times \frac{42}{2}$$
$$= 10794$$

4 **a** $1 + 5 + 9 + 13 + 17 + 21 + 25 + 29 + 33 + 37$

b $\sum_{i=1}^{10} (4i - 3)$

c $S_n = \left(2u_1 + (n-1)d\right) \times \frac{n}{2}$

$$S_{10} = \left(2(1) + (10-1)(4)\right) \times \frac{10}{2}$$
$$= 190$$

$$S_n = \left(2(1) + (n-1)(4)\right) \times \frac{n}{2}$$
$$= (4n-2) \times \frac{n}{2}$$
$$= n(2n-1)$$

d $n(2n-1)=2000$
$2n^2-n=2000$
$2n^2-n-2000=0$
$n=31.9$

It will take Janet 32 years to have a total of 2000 acres.

5 **a** $S_n=\frac{n}{2}\left(2a_1+d\left(n-1\right)\right)$
$a_1=S_1=4, d=-3$
$S_n=\frac{n}{2}\left(8-3\left(n-1\right)\right)$ or $S_n=\frac{n}{2}\left(11-3n\right)$

b $S_{10}=\frac{10}{2}\left(11-3\left(10\right)\right)$
$=5\left(11-30\right)$
$=5\left(-19\right)$
$=-95$

c $-250>\frac{n}{2}\left(11-3n\right)$
$-500>n\left(11-3n\right)$
$-500>-3n^2+11n$
$3n^2-11n-500>0$
$n=\frac{11}{6}\pm\frac{\sqrt{6121}}{6}$
only positive n
$n=14.873$
$n=15$

6 **a** $u_n=3+0.5(n-1)=0.5n+2.5$
$u_{10}=0.5(10)+2.5=7.5$ km

b $S_n=\left(2u_1+(n-1)d\right)\times\frac{n}{2}$

$S_{15}=\left(2(3)+(15-1)(0.5)\right)\times\frac{15}{2}$
$=97.5\text{km}$

7 **a** $u_1=22$
$d=3$
$n=29$

$S_{29}=\left(2(22)+(29-1)(3)\right)\times\frac{29}{2}$
$=1856$

b $S_n=\left(2u_1+(n-1)d\right)\times\frac{n}{2}$

$$\left(2(16)+(25-1)d\right)\times\frac{25}{2}\geq 6000$$
$$12.5(24d+32)\geq 6000$$
$$24d+32\geq 480$$
$$24d\geq 448$$
$$d\geq 18.7$$
$$d=19$$
$$12.5(24(19)+32)=6100$$

d = 19 will result in 6100 seats.

Exercise 4O

1 $\dfrac{4+3+4+2}{4}=3.25$

$d=3.25$

$u_1=27$

$u_n=27+3.25(n-1)=3.25n+23.75$

$3.25n+23.75=1800$

$3.25n=1776.25$

$n=546.5$

Tree is approximately 547 years old

2 $\dfrac{(-0.33)+(-0.38)+(-0.36)+(-0.39)+(-0.37)+(-0.33)+(-0.36)+(-0.35)+(-0.37)}{9}$

$=-0.36$

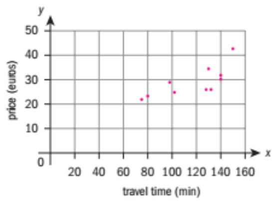
$u_n=2.28-0.36(n-1)=2.64-0.36n$

$2025\rightarrow u_{19}$

$u_{19}=2.64-0.36(19)=-4.2\%$

Exercise 4P

1 a A linear regression is appropriate because the data displays a roughly linear trend.

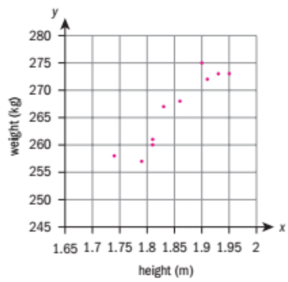


b $y=9.84+0.16x$ using GDC

c $y=9.84+0.16(120)=29.04$ euros

d 100 minutes, because it is interpolation (within the data set). Predicting for 10 minutes would be extrapolation beyond the data set.

2 a A linear regression is appropriate because the data display a roughly linear trend.



b $y = 93.7x + 92.8$

c $y = 93.7(1.8) + 92.8 = 261\,\text{kg}$

d A 1 cm increase in height corresponds to a 0.937 kg increase in weight.

3 a $f_1(x)$ is graph A because it has the lower y-intercept (8.11). $f_2(x)$ is Graph B.

b

x	1.2	6.5	3.7
y	7.4	2.8	4.3
$f_1(x)$	7.0768	2.5135	4.9243
residual	−0.3232	−0.2865	0.6243

$SSR_1 = (-0.3232)^2 + (-0.2865)^2 + (0.6243)^2 = 0.576$

x	1.2	6.5	3.7
y	7.4	2.8	4.3
$f_2(x)$	7.2044	2.3655	4.9219
residual	−0.1956	−0.4345	0.6219

$SSR_2 = (-0.1956)^2 + (-0.4345)^2 + (0.6219)^2 = 0.614$

Since $f_1(x)$ has the smaller sum of square residuals, it is the least squares regression equation.

Exercise 4Q

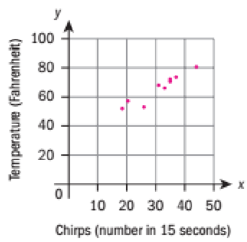
1 a The correlation coefficient is weak, so prediction from a linear regression will not be accurate.

b The data does not display a linear trend; it has a curved trend. A linear regression is not appropriate.

c A linear regression is appropriate here, but Kiernan is predicting the independent variable given a value of the dependent variable. Only predictions of the dependent variable are valid

d A linear regression is appropriate here, but Kiernan is extrapolating beyond the range of the data (x = 80 is outside the range of the data).

2 a The graph displays a linear trend.



b From GDC, $r = 0.951$; this is a strong positive correlation.

c Yes, it is a linear trend with a strong correlation.

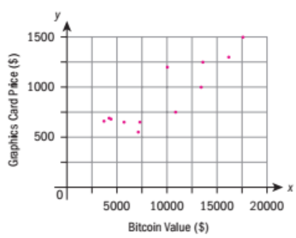
d i From GDC, $T(c) = 30.3 + 1.14c$

ii This is invalid, as we can use this regression to predict only the temperature from a known number of chirps.

iii $T(40) = 30.3 + 1.14(40) = 76°\text{F}$. Invalid, can use this regression to predict only the temperature from a known number of chirps.

e Answers may vary. Since c is for a 15-second period, by rounding 14 seconds to 15, the Almanac's formula becomes $T_1(c) = c + 40$. The gradients of the two equations are similar, and the y-intercepts differ by 10°F. $T_1(40) = 80°\text{F}$, which is 4°F different from the regression prediction.

3 a The data follows a linear trend, as evidenced by the scatter plot:



b From GDC, $r = 0.884$; this is a strong positive correlation

c Yes, it is a linear trend with a strong correlation.

d i From GDC, $G(b) = 342 + 0.06b$

ii $G(12320) = 342 + 0.06(12320) = \$1081.2$

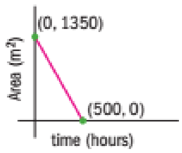
e A gradient of 0.06 means that for every dollar increase in Bitcoin value, graphics card prices increase by 6 cents.

Chapter Review

1 a $A(t) = 0$
$1350 - 2.7t = 0$
$2.7t = 1350$
$t = 500$ hours

b Domain: $\{t \mid 0 \le t \le 500)$
Range: $\{A \mid 0 \le A \le 1350\}$

c



d $A(5) = 1350 - 2.7(5) = 1336.5$; after 5 hours, there are 1336.5 m^2 of surface area remaining to be cleaned.

2 **a** function; not linear

b function; linear

c not a function: two cylinders with the same volume but different heights will have different radii.

3 **a** x = babies per woman and y = life expectancy. Equation from GDC: $y = -5.89x + 88.2$

b The scatter plot shows a linear relationship between the two variables; the correlation coefficient, $r = -0.726$, shows at least a moderately strong negative correlation.

c When $x = 2$, $y = 88.2 - 5.89(2) = 76.4$ years. This is a valid prediction because it is interpolation.

d Gradient of −5.89 in the regression equation means that an increase of 1 baby per woman corresponds to a decrease of 5.89 years of life expectancy.

4 **a** Graph 1, because it has a y-intercept of 0

b Graph 1: $\frac{150}{1800} = \frac{1}{12}$; the plane uses 12 litres to travel 1 km

Graph 2: $\frac{260-20}{3000-0} = \frac{240}{3000} = 0.08$; the plane takes 8 minutes to travel 100 km

c $T(d(x)) = 20 + 0.08\left(\frac{1}{12}x\right) = 20 + 0.0067x$

$T(150000) = 20 + 0.0067(150000) = 1025$ minutes or 17 hours 5 minutes

5 **a** $u_0 = 400$
$d = 150$
$u_n = 550 + 150(n-1) = 150n + 400$
$150n + 400 = 9500$
$150n = 9100$
$n = 60.67$

61 months

b $400 + 12(150) = \$2200$

$9500 - 2200 = \$7300$ remaining

$0 + k + 2k + 3k + 4k + \ldots + 11k = 7300$
$(0 + 11k)\left(\frac{12}{2}\right) = 7300$
$66k = 7300$
$k = 110.61$

k = \$111

c $400 + 150n + \left(\frac{n}{2}\right)(2(0) + 111(n-1)) = 20000$

$800 + 300n + 111n^2 - 111n = 40000$

$111n^2 + 189n - 39200 = 0$

$n = 17.96$

18 months in total, so 6 more months.

6 a $C(d) = 0.06d + 49$

b i

Destination	Cost to drive	Cost to fly
Brussels	$C(420) = 49 + 0.06(420) = 74.20$	130
Hamburg	$C(940) = 49 + 0.06(940) = 105.40$	150
Paris	$C(1030) = 49 + 0.06(1030) = 110.80$	240

Brussels: drive (costs €74.20); Hamburg: fly (drive costs €105.40); Paris: drive (costs €110.80)

ii $74.20 + 150 + 110.80 = €335$

7 $I = P \times r \times n$

$P + I = P + 8\%P$

$I = 0.08P$

$0.08P = P \times r \times 5$

$0.08 = 5r$

$r = 0.016$

$5 \times 0.016 \times P = 3200$

$P = 40000$

Investment = €40000

Rate = 1.6%

8 a $0.15 \times 46605 + 0.205 \times (93208 - 46605) + 0.26 \times (122000 - 93208) = 24030.29$

b $0.15 \times 46605 = 6990.75$

$6990.75 + 0.205 \times (93208 - 46605) = 16544.37$

$16544.37 + 0.26 \times (144489 - 93208) = 29877.43$

$29877.43 + 0.29 \times (205842 - 144489) = 47669.80$

$$T(x) = \begin{cases} 0.15x & 0 \le x \le 46605 \\ 6990.75 + 0.205(x - 46605) & 46605 < x \le 93208 \\ 16544.37 + 0.26(x - 93208) & 93208 < x \le 144489 \\ 29877.43 + 0.29(x - 144489) & 144489 < x \le 205842 \\ 47669.80 + 0.33(x - 205842) & x > 205842 \end{cases}$$

c i One payment: $T(162000) + T(122000) = 29877.43 + 0.29(162000 - 144489) + 24030.29$
$= 58985.91$

Two payments: $2 \times T(142000) = 2 \times \left(16544.37 + 0.26(142000 - 93208)\right)$
$= 58460.58$

ii Ian should choose two payments; he will save CA\$525.33.

9 a $f^{-1}(x) = \begin{cases} 15-3x, & 4 \le x \le 7 \\ -\frac{1}{2}x+5, & x < 4 \end{cases}$

Domain: $\{x \mid x \le 7\}$

Range: $\{y \mid y \ge -6\}$

b Inverse function would be $f^{-1}(x) = \begin{cases} 15-3x, & 4 \le x \le 7 \\ \frac{1}{2}x+1, & x > 4 \end{cases}$

But for $4 \le x \le 7$, there are two possible values of $f^{-1}(x)$ and so it is not a function.

In later chapters you will learn about the horizontal line test which is a simple way to determine whether or not a function has an inverse.

Exam Style Questions

10 a i $m = \frac{140-100}{100-70} = \frac{4}{3}$

ii $c = y - mx = 100 - \frac{4}{3}(70) = 6\frac{2}{3}$

b m is positive, so correlation is positive

c $\bar{y} = \frac{4}{3}(90) + 6\frac{2}{3} = 127$

d $y = \frac{4}{3}(60) + 6\frac{2}{3} = 86.7$

11 a $70 + 8.35d < 30 + 12.5d$
$40 < 4.15d$
$9.64 < d$

Car-nage is cheaper for 10 days or more. Therefore, Abel's holiday is at least 10 days.

b $14 \le d \le 21$
$70 + 8.35(14) = 186.90$
$70 + 8.35(21) = 245.35$
$\{C \mid 186.90 \le C \le 245.35\}$

12 a i From GDC, $r = 0.849$

ii Strong positive

iii From GDC, $y = 0.24 + 0.94x$

b i From GDC, $r = 0.267$

ii Weak positive

iii Data does not show a strong enough linear correlation for a linear regression line to be valid.

13 a $b = \frac{0-5}{12-0} = -\frac{5}{12}$

$a = 5$

b $y = 5 - \frac{5}{12}x$

$\frac{5}{12}x = 5 - y$

$x = 12 - \frac{12}{5}y$

$f^{-1}(x) = 12 - \frac{12}{5}x$

c $5 - \frac{5}{12}x = 12 - \frac{12}{5}x$

$\frac{119}{60}x = 7$

$x = \frac{60}{17} = 3\frac{9}{17}$

d $h^{-1}(x)$ is a reflection of $h(x)$ on the line $y = x$ so they will intersect when $h(x) = x$.

14 a $(3a+b)-7 = (5a-6b)-(3a+b) = (2a+9b+4)-(5a-6b)$

$a + 8b = 7$

$6a - 14b = 11$

$a = 3$

$b = 0.5$

b Substituting values gives the sequence $7, 9.5, 12, 14.5, \ldots$

$u_1 = 7$ and $d = 2.5$

$S_n = \left(2u_1 + (n-1)d\right) \times \frac{n}{2} > 1000$

$\left(2(7) + 2.5(n-1)\right) \times \frac{n}{2} > 1000$

$n(11.5 + 2.5n) > 2000$

$2.5n^2 + 11.5n - 2000 > 0$

$n > 26.0776\ldots$

$n = 27$

15 a $\{f \mid f \in \ \} \ f(x) \in$

b $(g \circ f)(x) \geq 18$

c $(f \quad g \quad f)(x) = 2x^2 + 18 - 24 = 2x^2 - 6$

$2x^2 - 6 = 0$

$x = \pm 1.73 \ \left(\text{or } \pm\sqrt{3}\right)$

d

$$
\begin{aligned}
f^{-1}(x) &= x + 24 \\
\left(g \circ f \circ f^{-1}\right)(x) &= g(x) \\
\left(g \circ f\right)(x + 24) &= 2(x + 24)^2 + 18 \\
&= 2x^2 + 96x + 1170 \\
g(x) &= 2x^2 + 96x + 1170
\end{aligned}
$$

5 Measuring and analysing randomness

Skills check

1 **a** $P(\text{prime}) = \dfrac{n(\{2,5,11,17\})}{n(\{1,2,4,5,9,10,11,16,17,25,26,27\})} = \dfrac{4}{12} = \dfrac{1}{3}$

b $P(\text{odd}) = \dfrac{n(\{1,5,9,11,17,25,27\})}{n(\{1,2,4,5,9,10,11,16,17,25,26,27\})} = \dfrac{7}{12}$

c $P(\text{square number}) = \dfrac{n(\{1,4,9,16,25\})}{n(\{1,2,4,5,9,10,11,16,17,25,26,27\})} = \dfrac{5}{12}$

2 **a** $P(\text{female}) = \dfrac{n(\text{female})}{n(\text{total})} = \dfrac{6+51}{12+47+6+51} = \dfrac{57}{116}$

b $P(\text{male smoker}) = \dfrac{n(\text{male smoker})}{n(\text{total})} = \dfrac{12}{12+47+6+51} = \dfrac{12}{116} = \dfrac{3}{29}$

c $P(\text{non-smoker}) = \dfrac{n(\text{non-smoker})}{n(\text{total})} = \dfrac{47+51}{12+47+6+51} = \dfrac{98}{116} = \dfrac{49}{58}$

Exercise 5A

1 $P(\text{letter in "MATHS"}) = \dfrac{n(\{R,A,N,D,O,M\} \cap \{M,A,T,H,S\})}{n(\{R,A,N,D,O,M\})}$

$= \dfrac{n(\{A,M\})}{n(\{R,A,N,D,O,M\})} = \dfrac{2}{6} = \dfrac{1}{3}$

2 Let the area of each sector be A

a $P(\text{at least } 4) = \dfrac{17A}{20A} = \dfrac{17}{20}$

b $P(\text{more than } 6) = \dfrac{14A}{20A} = \dfrac{7}{10}$

c $P(\text{less than } 30) = \dfrac{20A}{20A} = 1$

d $P(\text{no more than } 14) = \dfrac{14A}{20A} = \dfrac{7}{10}$

e $P(\text{prime}) = \dfrac{n(\{2,3,5,7,11,13,17,19\})}{n(\{1,2,3,\ldots,18,19,20\})} = \dfrac{8}{20} = \dfrac{2}{5}$

f $P(\text{square}) = \dfrac{n(\{1,4,9,16\})}{n(\{1,2,3,\ldots,18,19,20\})} = \dfrac{4}{20} = \dfrac{1}{5}$

g $P(\text{solution to } x^2 = 3) = 0$ because the only solutions to $x^2 = 3$ are $x = \pm\sqrt{3}$

3 **a** $P(\text{travelled by car}) = \dfrac{40+33}{40+33+59+41+37+29} = \dfrac{73}{239}$

b $P(\text{male travelled by foot}) = \frac{37}{40+59+37} = \frac{37}{136}$

c Number of shoppers by bus $= 1300 \times \frac{59+41}{40+33+59+41+37+29} = 543.933 \approx 544$

4 **a** $P(\text{equal to 0000}) = \frac{n(\{0000\})}{n(\{0000, 0001, \ldots, 9998, 9999\})} = \frac{1}{10000}$

b $P(\text{less than 8000 and more than 7900}) = \frac{n(\{7901, \ldots, 7999\})}{n(\{0000, 0001, \ldots, 9998, 9999\})} = \frac{99}{10000}$

c $P(\text{divisible by 10}) = \frac{n(\{0000, 0010, 0020, \ldots, 9980, 9990\})}{n(\{0000, 0001, \ldots, 9998, 9999\})} = \frac{1000}{10000} = \frac{1}{10}$

d $P(\text{at least 13}) = \frac{n(\{0013, 0014, \ldots, 9999\})}{n(\{0000, 0001, \ldots, 9998, 9999\})} = \frac{9987}{10000}$

5 The theoretical probability can be calculated as $\frac{1}{4}$.

6 Predicted number of smokers $= 11278 \times 0.17 = 1917.26$

7 Expected number of correct answers $= 10 \times \frac{1}{5} = 2$

Exercise 5B

1 Draw a sample space diagram to show the products of the two numbers on the tetrahedral and the octahedral dice:

M	1	2	3	4	5	6	7	8
1	1	2	3	4	5	6	7	8
2	2	4	6	8	10	12	14	16
3	3	6	9	12	15	18	21	24
4	4	8	12	16	20	24	28	32

Hence:

a $P(M \text{ is odd}) = \frac{8}{32} = \frac{1}{4}$

b $P(M \text{ is prime}) = \frac{6}{32} = \frac{3}{16}$

c $P(M \text{ is odd and prime}) = \frac{n(M \text{ is odd and prime})}{n(U_1)} = \frac{4}{32} = \frac{1}{8}$

Draw a sample space diagram to show the products of the two numbers on the two cubical dice:

N	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	8	10	12
3	3	6	9	12	15	18
4	4	8	12	16	20	24
5	5	10	15	20	25	30
6	6	12	18	24	30	36

Hence:

d $P(N \text{ is odd}) = \frac{9}{36} = \frac{1}{4}$

e $P(N > 13) = \frac{13}{36}$

f $P(N \text{ is factor of } 36) = \frac{20}{36} = \frac{5}{9}$.

g For example

Lots of trivial events with zero probability, e.g. $P(M > 40) = P(N > 40) = 0$

Lots of trivial events with probability 1, e.g. $P(M < 40) = P(N < 40) = 1$

$\text{P(M is even)} = \text{P(N is even)} = \frac{3}{4}$

$\text{P}(4 \le M \le 12) = \text{P}(4 \le N \le 12) = \frac{1}{2}$

$\text{P}(N \le 4) = \text{P}(M = 2, 4 \text{ or } 6) = \frac{1}{4}$

2 **a** From the Venn diagram: $127 - 41 - 29 - 52 = 5$

b $P(\text{only smartphone}) = \frac{n(\{\text{only smartphone}\})}{n(U)} = \frac{41}{127}$

c Estimated number $= 10000 \times \frac{52}{127} = 4094.488\ldots \approx 4094$

3 **a** We know that $20 - 2 = 18$ study either one or both of biology and history, therefore, the number that study both is $12 + 15 - 18 = 9$

$$P(\text{both biology and history}) = \frac{9}{20}$$

b $\frac{n(\{\text{biology and history}\})}{n(\{\text{biology}\})} = \frac{9}{12} = \frac{3}{4}$

4 Semi final - A vs B:

A \ B	3	3	3	3	3	3
4	A	A	A	A	A	A
4	A	A	A	A	A	A
4	A	A	A	A	A	A
4	A	A	A	A	A	A
0	B	B	B	B	B	B
0	B	B	B	B	B	B

$P(A \text{ beats } B) = \frac{n(A)}{36} = \frac{24}{36} = \frac{2}{3}$, $P(B \text{ beats } A) = 1 - P(A \text{ beats } B) = 1 - \frac{2}{3} = \frac{1}{3}$

Semi final - C vs D:

C \ D	1	1	1	5	5	5
2	C	C	C	D	D	D
2	C	C	C	D	D	D
2	C	C	C	D	D	D
2	C	C	C	D	D	D
6	C	C	C	C	C	C
6	C	C	C	C	C	C

$P(C \text{ beats } D) = \frac{n(C)}{36} = \frac{24}{36} = \frac{2}{3}$, $P(D \text{ beats } C) = 1 - P(D \text{ beats } C) = 1 - \frac{2}{3} = \frac{1}{3}$

5 It was show in in Q4 that A is likely to beat B. The following sample space diagram demonstrates the possible outcomes of B vs C

B \ C	2	2	2	2	6	6
3	B	B	B	B	C	C
3	B	B	B	B	C	C
3	B	B	B	B	C	C
3	B	B	B	B	C	C
3	B	B	B	B	C	C
3	B	B	B	B	C	C

$P(B \text{ beats } C) = \frac{n(B)}{36} = \frac{24}{36} = \frac{2}{3}$, $P(C \text{ beats } B) = 1 - P(B \text{ beats } C) = 1 - \frac{2}{3} = \frac{1}{3}$

The following sample space diagram demonstrates the possible outcomes of C vs A

A \ C	2	2	2	2	6	6
4	A	A	A	A	C	C
4	A	A	A	A	C	C
4	A	A	A	A	C	C
4	A	A	A	A	C	C
0	C	C	C	C	C	C
0	C	C	C	C	C	C

$P(C \text{ beats } A) = \frac{n(C)}{36} = \frac{20}{36} = \frac{5}{9}$, $P(A \text{ beats } C) = 1 - P(C \text{ beats } A) = 1 - \frac{5}{9} = \frac{4}{9}$

So therefore, A is likely to beat B which is likely to beat C which in turn is likely to beat A.

6

	1	**2**	**3**	**4**	**5**	**6**
1	1	2	3	4	5	6
2	2	2	3	4	5	6
3	3	3	3	4	5	6
4	4	4	4	4	5	6
5	5	5	5	5	5	6
6	6	6	6	6	6	6

$P(\text{at most } 4) = \frac{n(\{1,2,3,4\})}{36} = \frac{16}{36} = \frac{4}{9}$

Exercise 5C

1 **a** Independent because a fair coin has nothing to do with a die

b Neither because rain today may be indicative of rain tomorrow

c Not mutually exclusive because 2 is both prime and even. In addition, $\mathrm{P}(D) = \frac{3}{6} = \frac{1}{2}$ but

$\mathrm{P}(D \mid E) = \frac{1}{3}$ so not independent (this can also be shown by the fact

$\mathrm{P}(D \cap E) = \frac{1}{2} \neq \frac{3}{6} \times \frac{3}{6} = \mathrm{P}(D) \times \mathrm{P}(E)$)

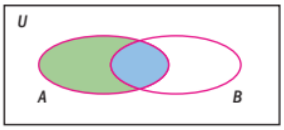
d Independent because one die does not impact another

e Mutually exclusive because if H occurs then C cannot occur

f Neither because $P(M) \times P(H) = 0.5 \times 0.7 \neq 0.2 = P(M \cap H)$

g Independent because $P(S) \times P(T) = \frac{10}{40} \times \frac{28}{40} = \frac{7}{40} = P(S \cap T)$, We note that the probability of the chosen student being a Spanish speaker is unchanged by the information he/she is a Turkish speaker, $\mathrm{P}(S) = \frac{10}{40} = \frac{1}{4}$, $\mathrm{P}(S \mid T) = \frac{7}{28} = \frac{1}{4}$

2



a $(A \cap B)$ is shaded in blue and $(A \cap B')$ is shaded in green. The total shaded area is equivalent to A. The two sets are clearly mutually exclusive as the areas shaded do not overlap.

b From the equivalence $A = (A \cap B) \cup (A \cap B')$ we can deduce that

$$\mathrm{P}(A) = \mathrm{P}((A \cap B) \cup (A \cap B'))$$

c Applying the probability law for union of mutually exclusive events

$$\mathrm{P}(A) = \mathrm{P}((A \cap B) \cup (A \cap B')) = \mathrm{P}((A \cap B)) + \mathrm{P}((A \cap B'))$$ by applying the law for independent events, $$\mathrm{P}(A) = \mathrm{P}(A) \times \mathrm{P}(B) + \mathrm{P}((A \cap B'))$$

d Rearranging the result from **c** we get

$$\mathrm{P}((A \cap B')) = \mathrm{P}(A) - \mathrm{P}(A) \times \mathrm{P}(B) = \mathrm{P}(A)(1 - \mathrm{P}(B))$$

Hence if A and B are independent then so are A and B'.

3 **a** **i** $P(4) \times P(1) \times P(3) \times P(5) \times P(2) = \left(\frac{1}{6}\right)^5 = \frac{1}{7776}$

ii $P(1) \times P(1) \times P(1) \times P(1) \times P(1) = \left(\frac{1}{6}\right)^5 = \frac{1}{7776}$

iii $P(6) \times P(5) \times P(4) \times P(3) \times P(2) = \left(\frac{1}{6}\right)^5 = \frac{1}{7776}$

b

$$\begin{aligned} P(\text{at least one 3}) &= 1 - P(\text{no threes}) \\ &= 1 - P(\text{not a three})^5 \\ &= 1 - \left(\frac{5}{6}\right)^5 \\ &= 1 - \frac{3125}{7776} \\ &= \frac{4651}{7776} \end{aligned}$$

c Expected number of times that sequence $1,1,1,1,1$ appears in 10000 trials is

$$10000\times\frac{1}{7776}=1.286$$

d $P(\text{Yahtzee}) = 6\times\left(\frac{1}{6}\right)^5 = \frac{1}{1296}$

e After $3\times1296=3888$ throws

4 **a** Because G and H are mutually exclusive, $P(G\cap H)=0$,

$$\begin{aligned}P(G\cup H)&=P(G)+P(H)-P(G\cap H)\\&=0.3+0.6-0=0.9\end{aligned}$$

b Because G and H are independent, $P(G\cap H)=P(G)\times P(H)=0.3\times0.6=0.18$,

$$\begin{aligned}P(G\cup H)&=P(G)+P(H)-P(G\cap H)\\&=0.3+0.6-0.18=0.72\end{aligned}$$

c $P(G\cap H)=P(G)+P(H)-P(G\cup H)=0.3+0.6-0.63=0.27$,

$$P(H\mid G)=\frac{P(H\cap G)}{P(G)}=\frac{0.27}{0.3}=0.9$$

5 **a** $P(\text{Barbora wins on one of her first two throws}) =$

$= P(\text{Barbora wins on her first throw}) + P(\text{Barbora wins on her second throw})$

$$=\frac{1}{8}+\left(\frac{7}{8}\times\frac{7}{8}\times\frac{1}{8}\right)=\frac{113}{512}=0.221$$

b If she went second, the probability of winning on her first two throws would be:

$P(\text{Barbora wins on one of her first two throws}) =$

$= P(\text{Barbora wins on her first throw}) + P(\text{Barbora wins on her second throw})$

$$=\left(\frac{7}{8}\times\frac{1}{8}\right)+\left(\frac{7}{8}\times\frac{7}{8}\times\frac{7}{8}\times\frac{1}{8}\right)=\frac{791}{4096}=0.193$$

0.193<0.221 so this indicates that Barbora has an advantage because she goes first.

To be sure Barbora has an advantage you need to add up the probabilities of Barbora winning on her first, second, third …. go to see if these reach above 0.5. In fact the probability she wins on one of her first 11 goes is 0.505 which indicates she definitely has the advantage.

6 $P(A\mid B)=\dfrac{P(A\cap B)}{P(B)}$.

Also, $P(A\cap B)=P(A)+P(B)-P(A\cup B)$. Combining these, we get

$$P(A \mid B) = \frac{P(A) + P(B) - P(A \cup B)}{P(B)}$$
$$= \frac{P(A)}{P(B)} + 1 - \frac{P(A \cup B)}{P(B)}$$
$$P(B) = \frac{P(A) - P(A \cup B)}{P(A \mid B) - 1}$$
$$= \frac{0.35 - 0.75}{0.35 - 1}$$
$$= \frac{8}{13}$$

Exercise 5D

1 $P(\text{same colour}) = P(\text{both green}) + P(\text{both yellow}) + P(\text{both red})$
$$= \frac{12}{27} \times \frac{11}{26} + \frac{8}{27} \times \frac{7}{26} + \frac{7}{27} \times \frac{6}{26} = \frac{230}{702} = \frac{115}{351}$$

2 $P(\text{different metals}) = 1 - P(\text{same metals})$
$$= 1 - \frac{13 \times 13 + 10 \times 10 + 12 \times 12}{(13 + 10 + 12)^2}$$
$$= 1 - \frac{413}{1225}$$
$$= 1 - \frac{59}{175}$$
$$= \frac{116}{175}$$

3 **a** $P(\text{passes QCI}) = 0.7 \times 0.9 + 0.3 \times 0.95 = 0.915$

b $P(\text{supplier D} \mid \text{passes QCI}) = \dfrac{P(\text{passes QCI} \mid \text{supplier D}) \times P(\text{supplier D})}{P(\text{passes QCI})}$
$$= \frac{0.95 \times 0.3}{0.915} = 0.3115$$

c Expected number to fail QCI is $2000 \times (1 - 0.915) = 170$

d Let x be the proportion of strawberries from supplier D,
$P(\text{passes QCI}) = 0.93 = (1 - x) \times 0.9 + x \times 0.95 \Rightarrow x = 0.6$

4 Probability of a double 6 in any throw is $\dfrac{1}{36}$.

$P(\text{at least one double 6}) = 1 - P(\text{no double 6})$
$$= 1 - \left(\frac{35}{36}\right)^{24} = 0.4914$$

5 $P((B \cap F)') = 1 - (P(B \cap F)) = 1 - (P(B) + P(F) - P(B \cup F))$
$$= 1 - (0.5 + 0.3 - 0.6) = 0.8$$

6 **a** $P(\text{all different birthdays}) = \dfrac{364 \times 363 \times 362 \times 361 \times 360 \times 359 \times 358 \times 357 \times 356 \times 355}{365^{10}} = 0.859$

b For n people,

$$\begin{aligned} P(\text{at least one shared birthday}) &= 1 - P(\text{all different birthdays}) \\ &= 1 - \frac{365!}{365^n(365-n+1)!} \\ &> 0.5 \end{aligned}$$

By trial and error, for 23 people:

$$\begin{aligned} P(\text{at least one shared birthday}) &= 1 - \frac{365 \times 364 \times \quad \times 344 \times 343}{365^{23}} \\ &= 0.5073 \end{aligned}$$

Chapter Review

1 As B and C are independent, $P(B \cap C) = 0.1 = P(B) \times P(C)$ and $P(B \cap C') = 0.4 = P(B) \times (1 - P(C))$, together these give that $P(B) = 0.5$ and $P(C) = 0.2$, so:

$$\begin{aligned} P(B' \cup C) &= P(B') + P(C) - P(B' \cap C) \\ &= (1 - 0.5) + 0.2 - (1 - 0.5) \times 0.2 \\ &= 0.6 \end{aligned}$$

2 **a** Let $P(Y) = y$, then $P(X) = P(Y')P(X \mid Y') + P(Y)P(X \mid Y) \Rightarrow$

$$\frac{2}{3} = (1-y) \times \frac{7}{12} + y \times \frac{8}{10} \Rightarrow y = P(Y) = \frac{5}{13}$$

b X and Y are independent if $P(X) = P(X \mid Y)$, so they are not independent in this case

3 **a** $n = 1$:

$$P(\text{at least one six}) = 1 - P(\text{no sixes}) = 1 - \frac{5}{6} = \frac{1}{6}$$

$n = 2$: $P(\text{at least one six}) = 1 - P(\text{no sixes}) = 1 - \left(\frac{5}{6}\right)^2 = \frac{11}{36}$

n : $P(\text{at least one six}) = 1 - P(\text{no sixes}) = 1 - \left(\frac{5}{6}\right)^n$

b $0.995 = 1 - \left(\frac{5}{6}\right)^n \Rightarrow 0.005 = \left(\frac{5}{6}\right)^n \Rightarrow n = \dfrac{\log(200)}{\log\left(\frac{6}{5}\right)} = 29.0603 \Rightarrow n = 30$

4 **a** $P(\text{seed grows}) = 0.65 \times 0.85 + 0.35 \times 0.74 = 0.8115$

b $P(\text{green and grows}) = P(\text{green}) \times P(\text{grows} \mid \text{green}) = 0.65 \times 0.85 = 0.5525$

c
$$\begin{aligned} P(\text{red or grows}) &= P(\text{red}) + P(\text{grows}) - P(\text{red and grows}) \\ &= 0.35 + 0.8115 - 0.35 \times 0.74 = 0.9025 \end{aligned}$$

5 **a** For $n = 1$, there are 3 pieces of paper, for $n = 3$ there are 8 pieces of paper, for general n there are $\dfrac{1+5n}{2}$ pieces of paper

b The only numbers that are divisible by 5 on the pieces of paper are 5, 15, 25, … For $n = 1$:

$$P(\text{divisible by 5}) = \frac{n(\{5\})}{3} = \frac{1}{3}$$

$$n = 3: \; P(\text{divisible by 5}) = \frac{n(\{5,15\})}{8} = \frac{2}{8} = \frac{1}{4}$$

$$n: \; P(\text{divisible by 5}) = \frac{\frac{n+1}{2}}{\frac{1+5n}{2}} = \frac{1+n}{1+5n}$$

6 $P(\text{two blue cars}) = \frac{1}{7} = \frac{6}{6+n} \times \frac{5}{5+n} \Rightarrow n = 9$

7 a

		Chromosome inherited from mother	
		X	X
Chromosome inherited from father	X	XX	XX
	Y	XY	XY

b $P(\text{female}) = \frac{n(\{XX, XX\})}{n(\{XX, XX, XY, XY\})} = \frac{2}{4} = \frac{1}{2}$

Exam style questions

8 a $P(\text{one contains nuts}) = \frac{3}{16} \times \frac{13}{15} + \frac{13}{16} \times \frac{3}{15} = \frac{13}{40}$

b $P(\text{at least one contains nuts}) = P(\text{one contains nuts}) + P(\text{two contain nuts})$

$$= \frac{13}{40} + \frac{3}{16} \times \frac{2}{15} = \frac{7}{20}$$

9 a $P(\text{all three green}) = 0.7 \times 0.4 \times 0.8 = 0.224$

b $P(\text{only one green}) = 0.7 \times 0.6 \times 0.2 + 0.3 \times 0.4 \times 0.2 + 0.3 \times 0.6 \times 0.8 = 0.252$

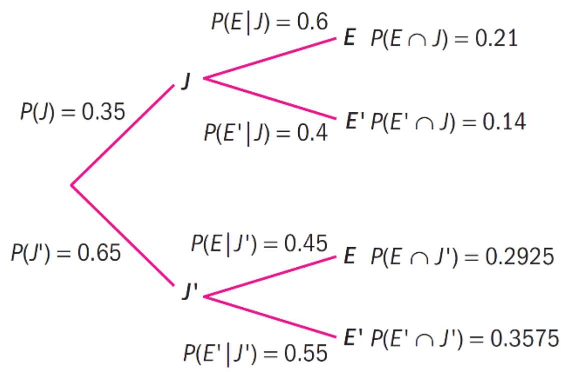
c Because of independence (as assumed in the question)

$$P(\text{lights 2 and 3 are green} \mid \text{light 1 is red}) = P(\text{lights 2 and 3 are green})$$
$$= 0.4 \times 0.8 = 0.32$$

d $P(\text{at least one is green}) = 1 - P(\text{all lights are red})$

$$= 1 - 0.3 \times 0.6 \times 0.2 = 0.964$$

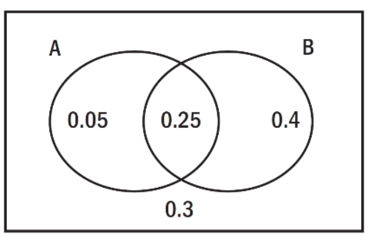
10 a Let J be the event that Jake solves it and E be the event that Elisa solves it



b $P(\text{at least one solves it}) = 1 - P(\text{neither solves it}) = 1 - 0.3575 = 0.6425$

c $P(J \mid E) = \dfrac{P(E \mid J)P(J)}{P(E)} = \dfrac{0.6 \times 0.35}{0.21 + 0.2925} = 0.41791$

11 $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$ hence $\mathrm{P}(A \cap B) = 0.3 + 0.65 - 0.7 = 0.25$

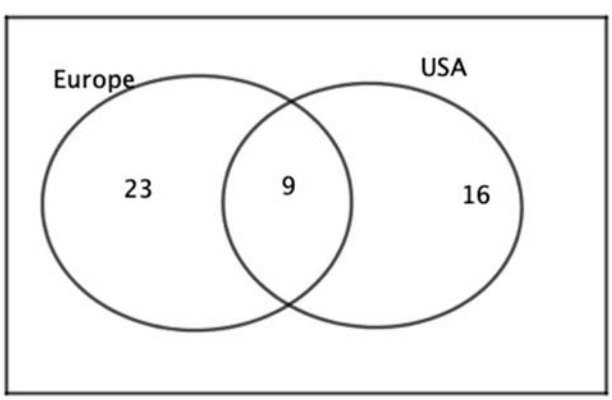


a $P(A' \cap B) = 0.4$

b $P(A \cup B') = 0.3 + 0.3 = 0.6$

c $P((A \cap B)') = 0.4 + 0.05 + 0.3 = 0.75$

12



a $n(\text{both USA and Europe}) = 9$

b $P(\text{Europe but not USA}) = \dfrac{23}{48}$

c Because $P(\text{Europe}) \times P(\text{USA}) = \dfrac{32}{48} \times \dfrac{25}{48} = \dfrac{25}{72} \neq \dfrac{9}{48} = P(\text{both})$

13 a $P(\text{Chemistry or Biology}) = \dfrac{15+12+6+10}{5+15+12+6+10+2} = \dfrac{43}{50}$

b $P(\text{neither Physics nor Biology}) = \dfrac{12+2}{50} = \dfrac{7}{25}$

c $P(\text{Physics} \mid \text{Chemistry}) = \dfrac{P(\text{Physics and Chemistry})}{P(\text{Chemistry})} = \dfrac{15}{15+12+6} = \dfrac{5}{11}$

d $P(\text{Biology} \mid \text{Physics}) = \dfrac{P(\text{Biology and Physics})}{P(\text{Physics})} = 0$

e $P(\text{Physics} \mid \text{not Biology}) = \dfrac{P(\text{Physics and not Biology})}{P(\text{not Biology})} = \dfrac{20}{5+15+12+2} = \dfrac{10}{17}$

14 a $P(A \cap B) = P(A) \times P(B) = 0.3 \times 0.5 = 0.15$

b $P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.3 + 0.5 - 0.15 = 0.65$

c $P(B' \cap A) = 0.5 \times 0.3 = 0.15$

d Because of independence $P(B \mid A') = P(B) = 0.5$

15 a Events B and C are not independent because C is a subset of B so if C occurs then B must also have occurred by definition, so $P(B \cap C) = P(C) \neq P(B) \times P(C)$

b Events A and C are mutually exclusive as they do not overlap on the Venn diagram, meaning that it is not possible for them both to occur together

c If A and B are independent, then $P(A) \times P(B) = P(A \cap B)$:
$P(A) \times P(B) = 0.3 \times 0.28 = 0.084 \neq 0.12 = P(A \cap B)$ so not independent

d Events A' and C' are not mutually exclusive as they overlap on the Venn diagram, meaning that it is possible for them both to occur together

e $P(A \cap C') = P(A) = 0.3$

16 a $a = P(T \cap S) = P(T \mid S)P(S) = 0.8 \times 0.9 = 0.72$
$c = P(S \cap T') = P(S) - P(S \cap T) = 0.9 - 0.72 = 0.18$

b $b = P(T \cap S') = P(T) - P(S \cap T) = 0.8 - 0.72 = 0.08$
$d = P((T \cup S)') = 1 - P(T \cup S) = 1 - (P(T) + P(S) - P(T \cap S))$
$= 1 - (0.8 + 0.9 - 0.72) = 0.02$

c $P((T \cap S') \cup (S \cap T')) = 0.08 + 0.18 = 0.26$

d $P(T \mid S') = \dfrac{P(T \cap S')}{P(S')} = \dfrac{0.08}{0.1} = 0.8$

e $P(S' \mid T') = \dfrac{P(S' \cap T')}{P(T')} = \dfrac{0.02}{0.2} = 0.1$

f i They are not mutually exclusive as they overlap on the Venn diagram, i.e. there are people who play both Squash and Tennis

ii They are independent as $P(S \cap T) = 0.72 = 0.8 \times 0.9 = P(S) \times P(T)$

6 Modelling relationships with functions: power and polynomial functions

Skills check

1 $-1 = 2\times 2^2 - 3\times 2 + c \Rightarrow c = -3$

2 a $x_{1,2} = \dfrac{-(-1)\pm\sqrt{(-1)^2-4(1)(-6)}}{2(1)} = \dfrac{1\pm\sqrt{25}}{2} = \dfrac{1\pm 5}{2}; x_1 = 3, x_2 = -2$

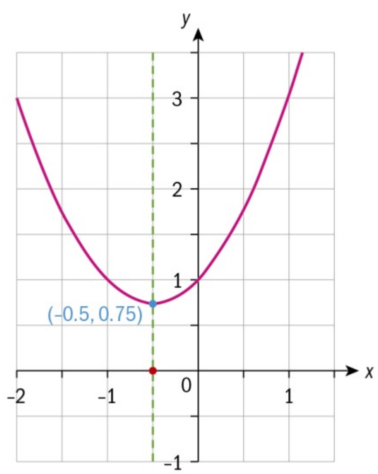
b $x_{1,2} = \dfrac{-1\pm\sqrt{(1)^2-4(3)(-2)}}{2(3)} = \dfrac{-1\pm\sqrt{25}}{6} = \dfrac{-1\pm 5}{6}; x_1 = \dfrac{2}{3}, x_2 = -1$

3 a $x^2+2x-8 = (x+4)(x-2) = 0 \Rightarrow x_1 = -4, x_2 = 2$

b $x^2-6x+8 = (x-4)(x-2) = 0 \Rightarrow x_1 = 4, x_2 = 2$

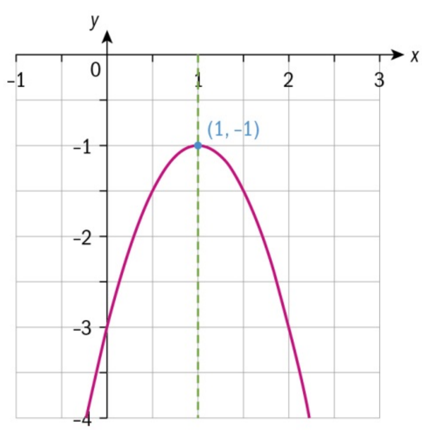
Exercise 6A

1 a



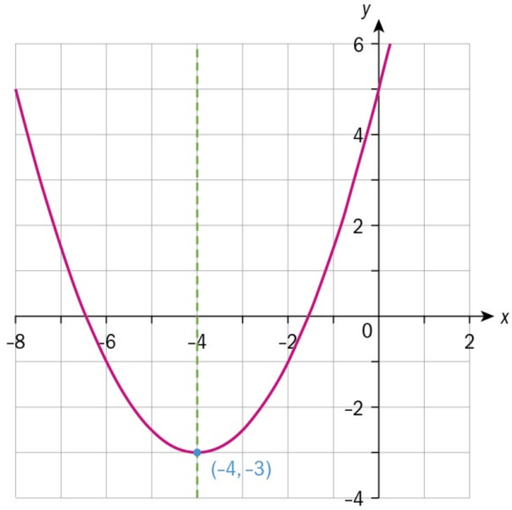
The axis of symmetry passes through the vertex and has equation x=-0.5.

b



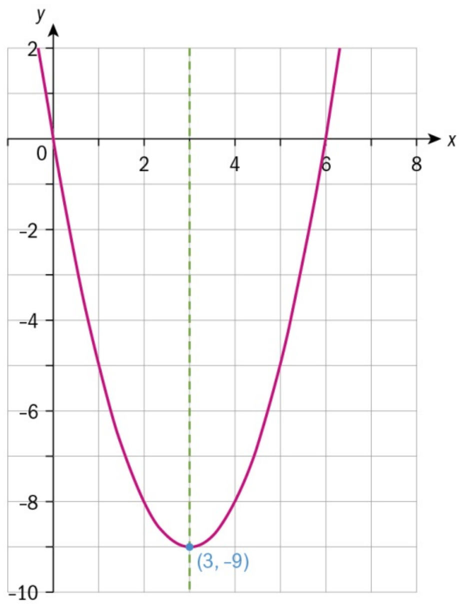
The axis of symmetry passes through the vertex and has equation x=1.

c



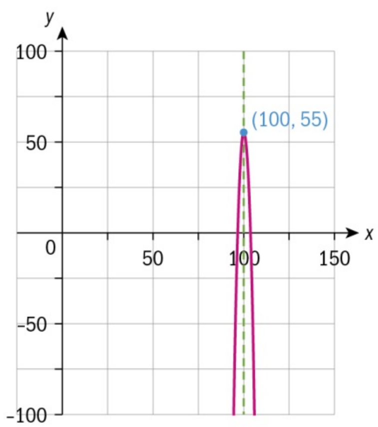
The axis of symmetry passes through the vertex and has equation x=-4.

d



The axis of symmetry passes through the vertex and has equation x=3.

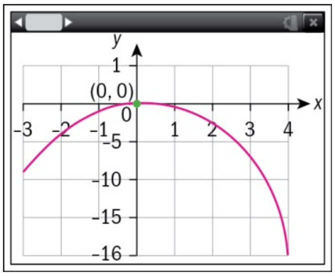
e



The axis of symmetry passes through the vertex and has equation $x=100$.

2

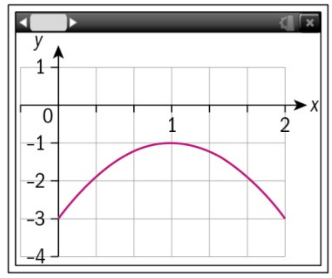
a



Vertex (0,0)

Range $-16 \le y \le 0$

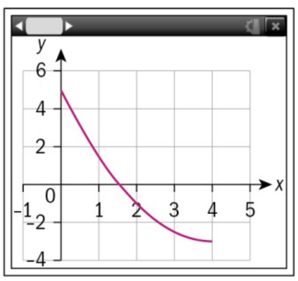
b



Vertex (1,-1)

Range $-3 \le y \le -1$

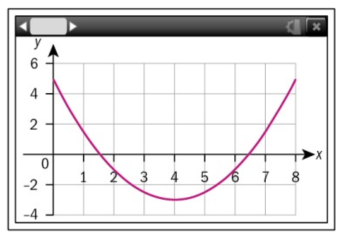
c



Vertex (4,-3)

Range $-3 \le y \le 5$

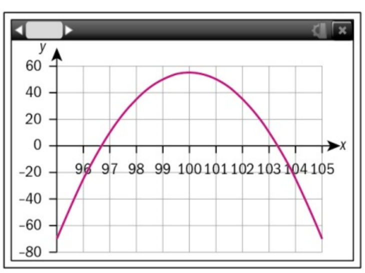
d



Vertex (4,-3)

Range $-3 \le y \le 5$

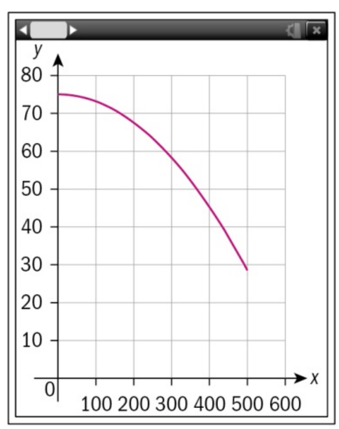
e



Vertex (100,55)

Range $-70 \le y \le 55$

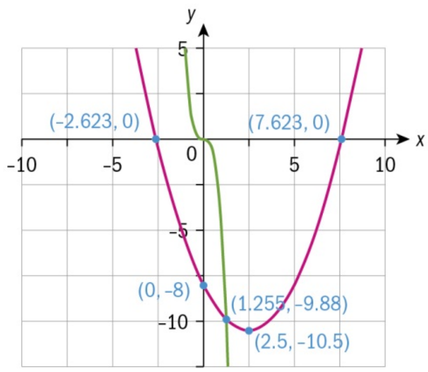
f



Vertex (0,75)

Range $28.5 \le y \le 75$

3



Draw the graph of $y = 0.4x^2 - 2x - 8$ using your GDC.

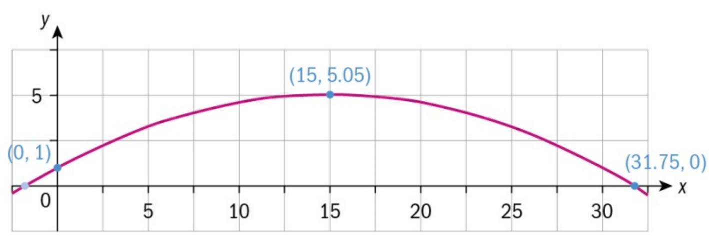
a From the GDC, graph crosses x-axis at (–2.62, 0) and (7.62, 0).

b From the GDC, graph intercepts y-axis at (0, –8).

c From the GDC, vertex of graph is (2.5, –10.5) so axis of symmetry is $x = 2.5$.

d Use your GDC to plot the graph of $y = -5x^3$ on the same axes. You can see that the graphs intersect at (1.25, –9.88).

4

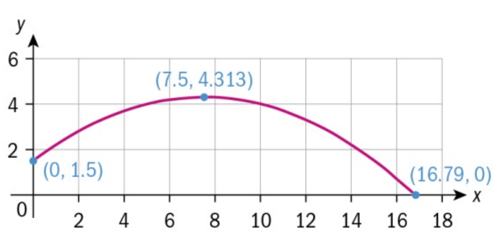


a From the GDC, ball reaches max. height of 5.05 m, as that is the y-coordinate of the vertex.

b From the graph, positive x-intercept is 31.75. This means that the ball has travelled 31.75 m when it lands on the ground.

c From the graph, y-intercept is (0, 1). This means that Zander hits the ball when it is at a height of 1 m above the ground.

5



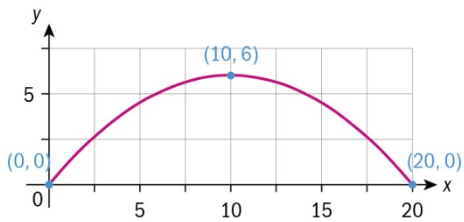
a From the GDC, max height is at the vertex of the graph, when $y = 4.31$. So max height is 4.31 m.

b Axis of symmetry passes through the vertex. From the GDC, this is the line $x = 7.5$.

c From the GDC, the graph crosses the positive x-axis when $x = 16.8$. This represents the horizontal distance the shot-put has travelled when it hits the floor.

d From the GDC, the y-intercept is $y = 1.5$. This represents the height of the shot-put when it leaves Omar's hand.

6

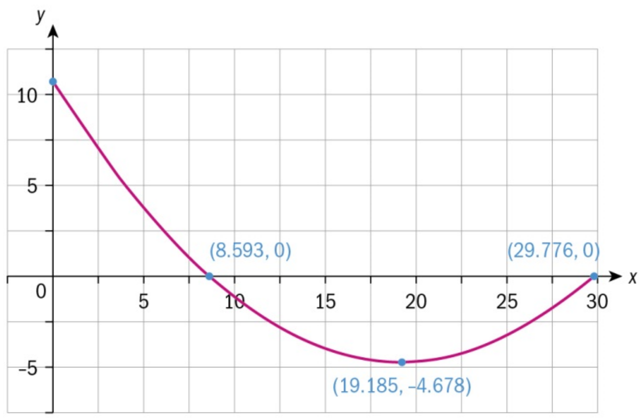


a From the GDC, max height is at the vertex of the graph, when $y = 6$. So max height is 6 m.

b Axis of symmetry passes through the vertex. From the GDC, this is the line $x = 10$.

c From the GDC, the graph crosses the x-axis when $x = 0$ and $x = 20$. $x = 0$ represents the position of the ball when Ziyue kicks it, and $x = 20$ is the horizontal distance the ball has travelled when it hits the ground.

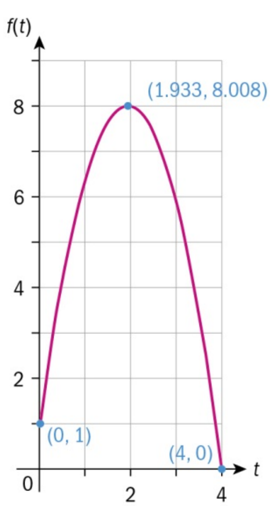
7



a From the GDC, the minimum depth is at the vertex. This occurs when $y = -4.68$, so minimum depth is 4.68 m below ground level.

b From the GDC, the x-intercepts at (8.59, 0) and (29.8, 0) represent the horizontal distance from the left-hand side of the ramp as it passes through ground level.

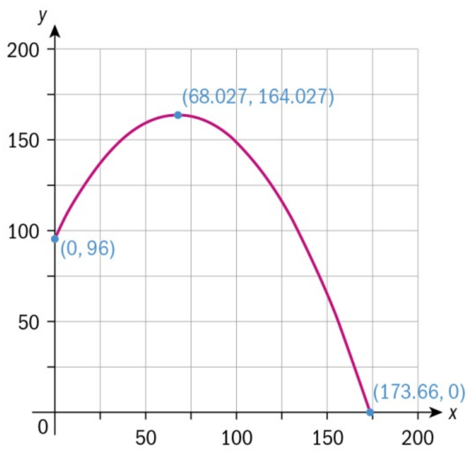
8



a From GDC, the maximum height is at the vertex where $y = 8.01$. Hence, max height is 8.01 m.

b Stone lands of the ground when $f(t) = 0$, i.e. when the graph crosses the positive x-axis. This occurs when $t = 4$ seconds.

9



a The bullet is fired from the top of the cliff, so the height of the cliff will be the same as the height of the bullet when $x = 0$ (that is, the y-value when $x = 0$). From GDC, height of cliff is 96 m.

b The bullet reaches its maximum height at the vertex of the graph, when $y = 164$. So max height is 164 m.

c Bullet hits the water when the graph crosses the positive x-axis. At this point, $x = 174$. So bullet hits the water when it is 174 m from the foot of the cliff.

10 a For an arithmetic sequence with first term a and common difference d, the sum of the first n terms is given by $S_n = \frac{n}{2}\left(2u_1 + (n-1)d\right)$.

Here, $u_1 = 13$ and $d = 9 - 13 = -4$ so

$$\begin{aligned} S_n &= \frac{n}{2}\left(2(13) + (n-1)(-4)\right) \\ &= \frac{n}{2}(26 - 4n + 4) \\ &= 15n - 2n^2 \end{aligned}$$

b

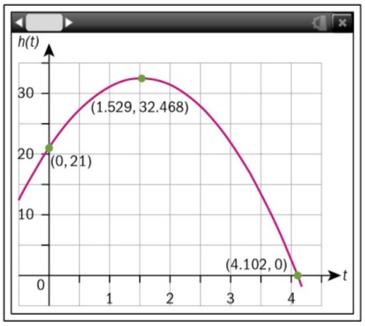
n	1	2	3	4	5	6	7	8
$S_n = 15n - 2n^2$	13	22	27	28	25	18	7	-8

c From table, max value of S_n is 28, and occurs when $n = 4$.

d $S_7 = 7 > 0$ and $S_8 = -8 < 0$ so 7 is the greatest value of n for which the sum is positive.

11 a $h_0 = 21,\ v_0 = 15$

so $h(t) = 21 + 15t - 4.905t^2$



b Max height is the y-value of the vertex of the graph. This is 32.5 m.

c Ball falls to the ground again when graph crosses the x-axis. This takes 4.10 seconds.

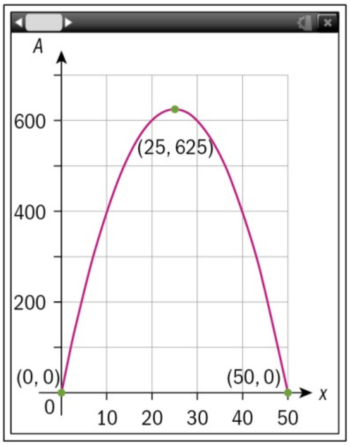
Exercise 6B

1 a Let y be the height of the frame. Then

$$\begin{aligned} 2x + 2y &= 100 \\ 2y &= 100 - 2x \\ y &= 50 - x \end{aligned}$$

b
$$\begin{aligned} A &= xy \\ &= x(50 - x) \\ &= 50x - x^2 \end{aligned}$$

c



d From GDC, x-intercepts are (0, 0) and (50, 0).

e The y-intercept is (0, 0).

f The equation of the axis of symmetry can be found from the vertex, or by finding the midpoint of the two x-intercepts: $x = \dfrac{0+50}{2} = 25$.

g From GDC, vertex is (25, 625).

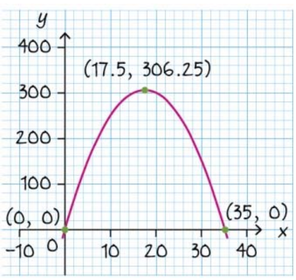
h Maximum area occurs at the vertex of the graph of A. Max area is 625 m^2, and occurs when $x = 25$ m and $y = 50 - 25 = 25$ m.

2 **a** Let y be the height of the frame. Then

$$\begin{aligned} 2x + 2y &= 70 \\ 2y &= 70 - 2x \\ y &= 35 - x \end{aligned}$$

b
$$\begin{aligned} A &= xy \\ &= x(35 - x) \\ &= 35x - x^2 \end{aligned}$$

c



d From the GDC, x-intercepts are at (0, 0) and (35, 0).

e These are the two values for the frame width which would give an area of zero. As such, these are the upper and lower limits for the width of the frame.

f The line of symmetry passes through the midpoint of the two x-intercepts, so

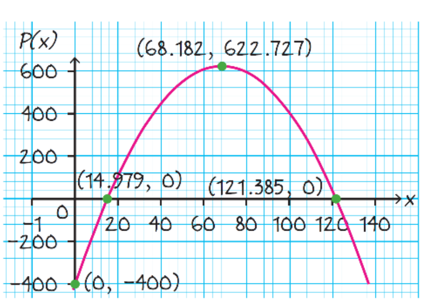
$$x = \frac{0+35}{2} = 17.5$$

The axis of symmetry passes through the maximum point of the graph, so this tells you the value of x that gives maximum area of the frame.

3 **a** (Profit) = (income) – (cost)

$$\begin{aligned} P(x) &= (-0.12x^2 + 30x) - (0.1x^2 + 400) \\ &= (-0.12 - 0.1)x^2 + 30x - 400 \\ &= -0.22x^2 + 30x - 400 \end{aligned}$$

b



c The x-intercepts are the number of books that give a profit of €0.

d From GDC (or by finding midpoint of x-intercepts), axis of symmetry is $x = 68.2$. Since it is impossible to produce 0.2 books, this tells us that producing 68 books will maximise the profit.

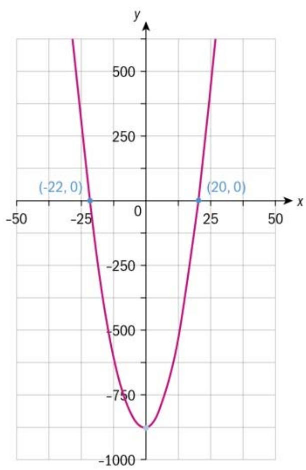
4 **a** $a = 6$ and $d = 10 - 6 = 4$ so

$$\begin{aligned} S_n &= \frac{n}{2}\left(2(6) + (n-1)(4)\right) \\ &= \frac{n}{2}(12 + 4n - 4) \\ &= 4n + 2n^2 \end{aligned}$$

b $880 = S_n = 4n + 2n^2$

$2n^2 + 4n - 880 = 0$

c If the sum of the first n terms is greater than 880, then the graph of $y = S_n - 880 = 2n^2 + 4n - 880$ is above the x-axis.



Since n can only be positive, $S_n > 880$ when $n > 20$.

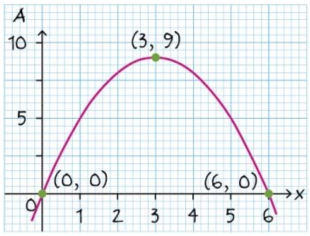
5 A parabola (graph of a quadratic function) is symmetric about its vertex. So the dolphin will fall back into the water at $2 \times 1.3 = 2.6$ s.

6 **a** **i** Let y be the length of the other side.

$$\begin{aligned} 2x + 2y &= 12 \\ 2y &= 12 - 2x \\ y &= 6 - x \end{aligned}$$

ii
$$\begin{aligned} A &= xy \\ &= x(6 - x) \\ &= 6x - x^2 \end{aligned}$$

iii



iv By looking at the vertex of the graph, maximum area for this design is 9 m^2.

b **i** Let x be the length of the vertical side in the diagram.

Then $2x + y = 12$

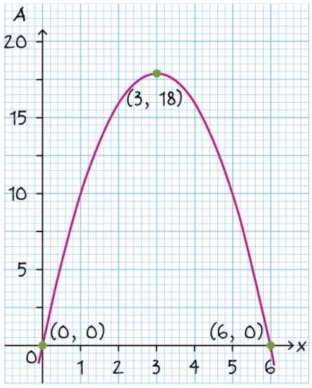
$y = 12 - 2x$

ii $A = xy$

$= x(12 - 2x)$

$= 12x - 2x^2$

iii



iv By looking at the vertex of the graph, maximum area for this design is 18 m^2.

c **i** $x + y = 12$

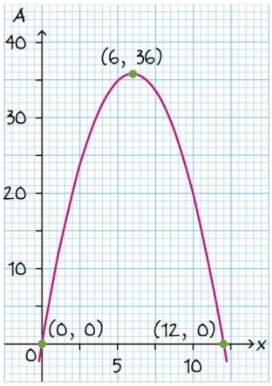
$y = 12 - x$

ii $A = xy$

$= x(12 - x)$

$= 12x - x^2$

iii



iv By looking at the vertex of the graph, maximum area for this design is 36 m^2.

d The design in part c will give the maximum area.

Exercise 6C

1 a i $f(x) = x^2 + 4$ has a line of symmetry at $x = 0$, so restricting the domain to $x \geq 0$ means the function is invertible. Hence $a = 0$

ii Let $y = x^2 + 4$. Then $x = \sqrt{y - 4}$

So, the inverse function is $f^{-1}(x) = \sqrt{x - 4}$ with domain $x \geq 4$ and range $y \geq 0$.

(Answers may vary.)

b i $f(x) = (x-1)^2 - 2$ has a line of symmetry at $x = 1$, so restricting the domain to $x \geq 1$ means the function is invertible. Hence $a = 1$

ii Let $y = (x-1)^2 - 2$. Then $\sqrt{y+2} = x - 1 \Rightarrow x = 1 + \sqrt{y+2}$

So inverse function is $f^{-1}(x) = 1 + \sqrt{x+2}$ with domain $x \geq -2$ and range $y \geq 1$.

c i $f(x) = 2(x+3)^2$ has a line of symmetry at $x = -3$, so restricting the domain to $x \geq -3$ means the function is invertible. Hence $a = -3$

ii Let $y = 2(x+3)^2$. Then $\sqrt{\frac{y}{2}} = x + 3 \Rightarrow x = \sqrt{\frac{y}{2}} - 3$

So inverse function is $f^{-1}(x) = \sqrt{\frac{x}{2}} - 3$ with domain $x \geq 0$ and range $y \geq -3$.

d i $f(x) = 1 - 3(x-2)^2$ has a line of symmetry at $x = 2$, so restricting the domain to $x \geq 2$ means the function is invertible. Hence $a = 2$

Let $y = 1 - 3(x-2)^2$. Then $\sqrt{\frac{1-y}{3}} = x - 2 \Rightarrow x = 2 + \sqrt{\frac{1-y}{3}}$

So inverse function is $f^{-1}(x) = 2 + \sqrt{\frac{1-x}{3}}$ with domain $x \leq 1$ and range $y \geq 2$.

2 **a** The revenue is the total income the company receives through sales of plates.

Revenue (R) = quantity of plates manufactured (Q) × price per plate (p)

$R(p) = (175 - 3.5p)p = 175p - 3.5p^2$

b Cost = fixed cost + cost per plate

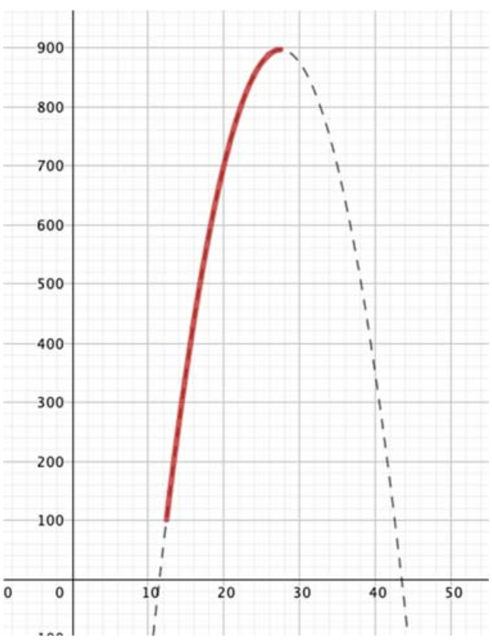
$$\begin{aligned} C(p) &= 875 + 5 \times Q(p) \\ &= 875 + 5(175 - 3.5p) \\ &= 1750 - 17.5p \end{aligned}$$

c Profit (P) = $R(p) - C(p)$

$$\begin{aligned} P(p) &= (175p - 3.5p^2) - (1750 - 17.5p) \\ &= -3.5p^2 + 192.5p - 1750 \end{aligned}$$

Maximum daily profit = €896.88 (when p=€27.5)

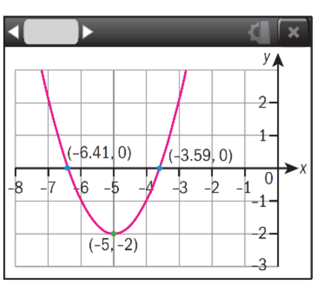
d Section of the curve P required is shown as red below



Domain of inverse function is range of original so $100 \le P \le 896.88$

Range of inverse is domain of original function so $12.45 \le p \le 28.5$

3 **a**

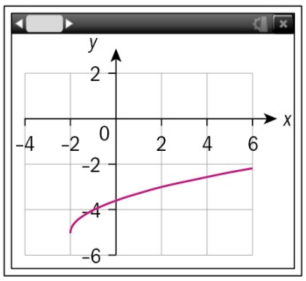


b $f(x)$ is not one-to-one if the domain is $x \in$. It fails the horizontal line test as it is symmetric about $x = -5$.

c Since axis of symmetry is $x = -5$, restricting the domain to $x \geq -5$ means that f is one-to-one and so invertible. (note: the answer $x \leq -5$ is also possible)

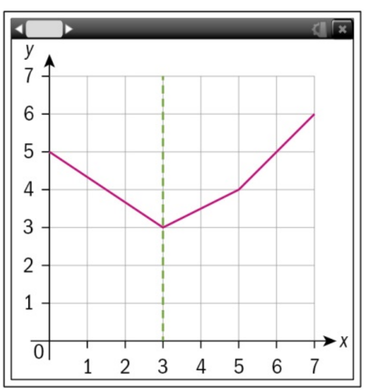
d Let $y = (x+5)^2 - 2$. Rearranging gives $\sqrt{y+2} - 5 = x$

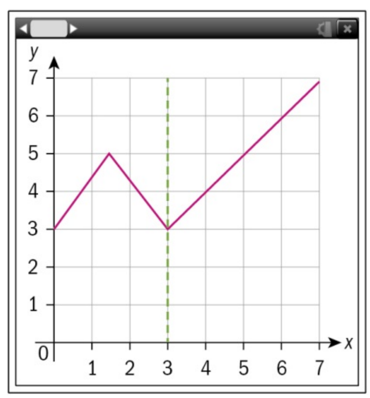
So $f^{-1}(x) = \sqrt{x+2} - 5$ with domain $x \geq -2$



Alternatively, use the table or point-plotting feature of your GDC to find several coordinates on the graph of $f(x)$. Invert these to find the corresponding coordinates of $f^{-1}(x)$ and hence plot the graph.

4 Function can have various shapes, including the following two:





Exercise 6D

Note: The questions in Exercise 6D can all be solved via two methods: By using quadratic regression or by simultaneous equation solver (both on GDC). These solutions use a mixture of the two methods.

1 Identify 3 points that the curve passes through, e.g. (–4, 0), (–2, –6) and (1, 0).

GDC quadratic regression gives $y = x^2 + 3x - 4$.

2 Identify 3 points that the curve passes through, e.g. (–1, 0), (0, 6) and (2, 0).

Let the quadratic equation be $y = ax^2 + bx + c$.

$(-1, 0) \Rightarrow 0 = a - b + c$
$(0, 6) \Rightarrow 6 = c$
$(2, 0) \Rightarrow 0 = 4a + 2b + c$

Solving simultaneously on GDC gives $y = -3x^2 + 3x + 6$.

3 Identify 3 points that the curve passes through, e.g. (–4, 0), (0, –4) and $\left(\frac{1}{2}, 0\right)$

GDC quadratic regression gives $y = 2x^2 + 7x - 4$.

4 Identify 3 points that the curve passes through, e.g. (2, 6), (3, 4) and (4, 6).

GDC quadratic regression gives $y = 2x^2 - 12x + 22$.

5 **a** Identify the approximate coordinates of 3 points that the curve passes through,

e.g. (–8.5, 0), (–5, 12) and (–1.5, 0).

GDC quadratic regression gives $y = -0.980x^2 - 9.80x - 12.5$.

(This is an approximate equation, since it is impossible to find three exact coordinates from the graph.)

b Identify the approximate coordinates of 3 points that the curve passes through, e.g. (0, 4), (1, 6) and (2, 4).

GDC quadratic regression gives $y = -2x^2 + 4x + 4$.

6 **a** Substituting $x = 3$ and $f(x) = -4$ into the function $f(x) = ax^2 + bx + c$ gives

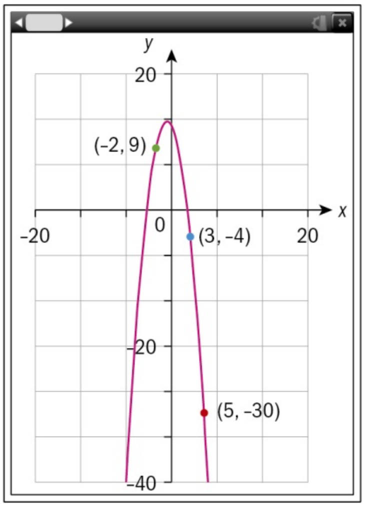
$$-4 = a \times 3^2 + b \times 3 + c$$
$$\Rightarrow 9a + 3b + c = -4$$

b Substituting $(-2, 9) \Rightarrow 9 = 4a - 2b + c$

Substituting $(5, -30) \Rightarrow -30 = 25a + 5b + c$

c Using simultaneous equation solver on GDC to solve the 3 equations from parts a and b gives $a = -\frac{52}{35}, b = -\frac{39}{35}, c = \frac{89}{7}$.

d



7 Since the arch is 192 m wide, the ends of the arch pass through the points with coordinates (–96,0) and (96,0).

Since the arch is 192 high, the vertex of the arch is at the point (0, 192).

Using quadratic regression with these three points, the equation of the arch is $y = -\frac{1}{48}x^2 + 192$

8 The ball is kicked from the origin, so passes through (0, 0).

Assuming the ball passes just over the head of the tallest player in the wall, it passes through (9.1, 2).

Assuming that the ball passes just under the crossbar of the goal, it passes through (20, 2.4).

Using quadratic regression with these three points, the optimum path for the ball is $y = -0.00915x^2 + 0.303x$.

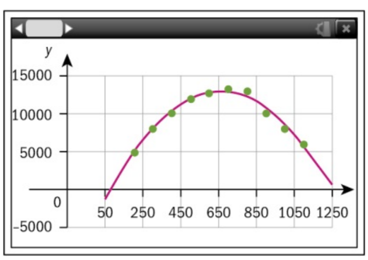
9 $f(x) = -0.12x^2 + 1.92x + 3$ has a vertex at $(8, 10.68)$ and passes through $(0, 3)$ and $(16, 3)$

New curve will have a vertex at $(8, 9.68)$ and will pass through $(0, 3)$ and $(16, 3)$

Equation is $f(x) = -0.104x^2 + 1.67x + 3$

Exercise 6E

1 a

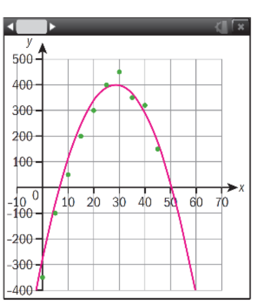


b Using quadratic regression on GDC, the best fit quadratic curve is $y = -0.0382x^2 + 50.5x - 3744$.

c $R^2 = 0.98$ which is a very strong quadratic association, and hence the equation $y = -0.0382x^2 + 50.5x - 3744$ is a good fit for this data.

d This equation only shows the association between the number of units sold and the profit. It is not linked to a particular period of the year, so could not be used to predict the company's profits at a particular time of year.

2 a

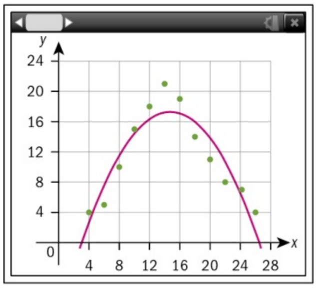


b Using quadratic regression on GDC, the best fit quadratic curve is $y = -0.895x^2 + 52.2x - 359$.

c $R^2 = 0.988$ which is a very strong quadratic association, and hence the equation $y = -0.895x^2 + 52.2x - 359$ is a good fit for this data.

d Week 52 lies outside the range of data given. Using this data to make a prediction for week 52 would be extrapolation, and therefor unreliable.

3 a

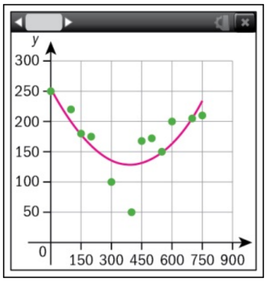


b Using quadratic regression on GDC, the best fit quadratic curve is $y = -0.126x^2 + 3.72x - 10.2$.

c $y(17) = -0.126(17)^2 + 3.72(17) - 10.2 = 16.7\,^{\circ}\text{C}$

Since $R^2 = 0.85$ suggests a very strong quadratic association, and because 17:00 is within the data range so is interpolation, this is likely to be a reliable estimate.

4 a

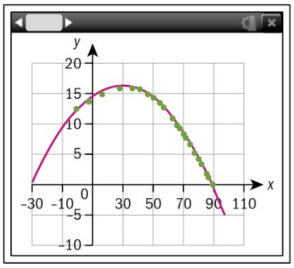


b Using quadratic regression on GDC, the best fit quadratic curve is $y = (8.293 \times 10^{-4})x^2 - 0.652x + 257$.

c A coefficient of determination of $R^2 = 0.61$ suggests only a moderate correlation. Cindy's data clearly contains errors as, for example, the point (550,150) is below (500, 172). The quadratic model is unlikely to be a good fit.

d Even though 410m is within the range of the data the model is unlikely to provide a good estimate.

5 a Plot the points on the scatter diagram. Then, using quadratic regression on GDC, the best fit quadratic curve is $y = -0.00462x^2 + 0.275x + 12.0$.



b The curve $y = -0.00462x^2 + 0.275x + 12.0$ has a maximum at (29.8, 16.1). So the optimal angle is 29.8°.

c The coefficient of determination is 0.998, which indicates the data almost perfectly follows a quadratic model. Therefore, it is entirely appropriate to use the model function found in part **a**.

Exercise 6F

1 a $x^2 \to \left(\frac{x}{2}\right)^2$ is a horizontal stretch, scale factor 2

$\left(\frac{x}{2}\right)^2 \to \left(\frac{x}{2}\right)^2 - 3$ is a vertical translation of 3 units down

b $2(x-3)^2 + 4 \to 2(x-3)^2$ is a vertical translation down 4 units

$2(x-3)^2 \to \frac{1}{2}\left[2(x-3)^2\right] = (x-3)^2$ is a vertical stretch scale factor $\frac{1}{2}$

$(x-3)^2 \to ((x-3)+3)^2 = x^2$ is a horizontal translation 3 units to the left

c $x^2 \to (x-1)^2$ is a horizontal translation 1 unit to the right

$(x-1)^2 \to 4(x-1)^2$ is a vertical stretch, scale factor 4

$4(x-1)^2 \to 4(x-1)^2 + 2$ is a vertical translation of 2 units up

d $2(x-3)^2 + 4 \to 2(x-2)^2 + 4$ is a horizontal translation 1 unit to the left

$2(x-2)^2 + 4 \to -2(x-2)^2 - 4$ is a reflection in the x-axis

$-2(x-2)^2 - 4 \to -(x-2)^2 - 2$ is a vertical stretch, scale factor 0.5

$-(x-2)^2 - 2 \to -(x-2)^2 + 2$ is a translation of 4 up.

2 a Translating x^2 by 3 units to the left gives $(x+3)^2$

Reflecting $(x+3)^2$ in the x-axis gives $-(x+3)^2$

Stretching $-(x+3)^2$ vertically with scale factor 2 gives $-2(x+3)^2$

b Translating x^2 by 2 units to the left gives $(x+2)^2$

Stretching $(x+2)^2$ horizontally with scale factor 2 gives $\left(\frac{1}{2}x+2\right)^2$

c Stretching x^2 horizontally with scale factor 2 gives $\left(\frac{1}{2}x\right)^2$

Translating $\left(\frac{1}{2}x\right)^2$ by 2 units to the left gives $\left(\frac{1}{2}(x+2)\right)^2 = \left(\frac{x}{2}+1\right)^2$

d Translating x^2 by 1 unit up and 2 units to the left gives $(x+2)^2+1$

Reflecting $(x+2)^2+1$ in the y-axis gives $(2-x)^2+1$

3 **a** This is the equation of a concave-up parabola. So it will have a minimum point. Since $2(x-1)^2 \geq 0$ for all x, the minimum value of the graph is $y=5$ and this happens when $x=1$. So the vertex is (1, 5).

b $y(3) = 2(3-1)^2+5 = 2\times 4+5 = 13$ so (3, 13) lies on the curve.

c **i** Reflect in y-axis: $(1,5)\to(-1,5)$

Horizontal translation 3 units right: $(-1,5)\to(2,5)$

Vertical stretch with scale factor 2: $(2,5)\to(2,10)$

Vertex is now $(2,10)$

ii Reflect in y-axis: $(3,13)\to(-3,13)$

Horizontal translation 3 units right: $(-3,13)\to(0,13)$

Vertical stretch with scale factor 2: $(0,13)\to(0,26)$

(3, 13) is now (0, 26)

d Equation in vertex form is $y=a(x-h)^2+k$ where the vertex $(h,k)=(2,10)$ so $y=a(x-2)^2+10$. Point (0,26) is on the curve, so

$$
\begin{aligned}
26 &= a(0-2)^2+10\\
26 &= 4a+10\\
4a &= 16\\
a &= 4\\
y &= 4(x-2)^2+10
\end{aligned}
$$

e Reflecting in the y-axis, curve becomes

$$y = 2((-x)-1)^2+5 = 2(-x-1)^2+5$$

which can be written as $y=2(x+1)^2+5$

Horizontal translation 3 units right, curve becomes

$$y = 2\left(\left(x-3\right)+1\right)^2+5 = 2(x-2)^2+5$$

Vertical stretch with scale factor 2, curve becomes

$$y = 4(x-2)^2+10$$

4 **a** $x^2 \to (x-2)^2$ horizontal translation $\begin{pmatrix}2\\0\end{pmatrix}$

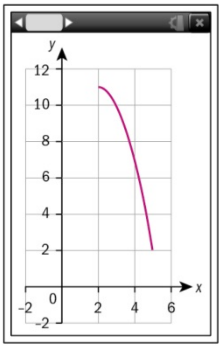
$(x-2)^2 \to -(x-2)^2$ reflection in the x-axis

$-(x-2)^2 \to 5-(x-2)^2$ vertical translation $\begin{pmatrix}0\\5\end{pmatrix}$

b Vertical stretch with scale factor 2: $5-(x-2)^2 \to 10-2(x-2)^2$

Translation $\begin{pmatrix}-1\\3\end{pmatrix}$: $10-2(x-2)^2 \to (10+3)-2((x+1)-2)^2 = 13-2(x-1)^2$

5 **a** **i**



ii Range $2 \le y \le 11$

iii $x^2 \to (x-2)^2$ horizontal translation $\begin{pmatrix}2\\0\end{pmatrix}$

$(x-2)^2 \to -(x-2)^2$ reflection in the x-axis

$-(x-2)^2 \to 11-(x-2)^2$ vertical translation $\begin{pmatrix}0\\11\end{pmatrix}$

b Translation $\begin{pmatrix}3\\-1\end{pmatrix}$ gives $11-(x-2)^2 \to (11-1)-((x-3)-2)^2 = 10-(x-5)^2$

Stretch with scale factor $\frac{1}{2}$ in y-direction gives $10-(x-5)^2 \to \frac{1}{2}\left[10-(x-5)^2\right] = 5-\frac{1}{2}(x-5)^2$

$g(x) = 5-\frac{1}{2}(x-5)^2$

6 Need to reverse the transformations, in the reverse order in which they were applied.

Translation by $\begin{pmatrix}-2\\-5\end{pmatrix}$ to take $h(x) \to g(x)$

Translation $\begin{pmatrix}0\\15\end{pmatrix}$ maps $2(x+2)^2-15 \to 2(x+2)^2$

Stretch with scale factor $\frac{1}{2}$ in y-direction maps $2(x+2)^2 \to (x+2)^2$

Translation $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ maps $(x+2)^2 \to ((x-2)+2)^2 = x^2$

7 **a** $x^2 \to (x-1)^2$ translation $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$(x-1)^2 \to 3(x-1)^2$ vertical stretch with scale factor 3

$3(x-1)^2 \to 3(x-1)^2 + 2$ translation $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$

b Translation $\begin{pmatrix} 0 \\ -3 \end{pmatrix}$ maps $3(x-1)^2 + 2 \to 3(x-1)^2 - 1$

Reflection in x-axis maps $3(x-1)^2 - 1 \to -3(x-1)^2 + 1$

So $h(x) = -3(x-1)^2 + 1$

8 **a**

$f(x) = x^2$

$(r\ \ f)(x) = f(x^2) = -(x^2)$ (reflection in the x axis)

$(t\ \ f\ \ r)(x) = t(-(x^2)) = 1 - x^2$ (vertical translation by 1)

b

$f(x) = x^2$

$(t\ \ f)(x) = t(x^2) = x^2 + 1$ (vertical translation by 1)

$(s\ \ t\ \ f)(x) = s(x^2+1) = 2(x^2+1) = 2x^2 + 2$ (vertical stretch by a factor of 2)

c

$f(x) = x^2$

$(s\ \ f)(x) = s(x^2) = 2(x)^2 = 2x^2$ (vertical stretch by a factor of 2)

$(t\ \ s\ \ f)(x) = t(2x^2) = 2x^2 + 1$ (vertical translation by 1)

d

$f(x) = x^2$

$(r\ \ f)(x) = r(x^2) = -(x^2)$ (reflection in the x axis)

$(s\ \ r\ \ f)(x) = s(-(x^2)) = -2(x^2) = -2x^2$ (vertical stretch by a factor of 2)

e Vertex of $f(x) = x^2$ is $(0,0)$

Part **a**: $(0,0) \to (0,0) \to (0,1)$

Part **b**: $(0,0) \to (0,1) \to (0,2)$

Part **c**: $(0,0) \to (0,0) \to (0,1)$

Part **d**: $(0,0) \to (0,0) \to (0,0)$

9 **a** **i** Translation $\begin{pmatrix} 3 \\ 0 \end{pmatrix}$: $(1,1) \to (4,1)$

Horizontal stretch with scale factor 2: $(4,1) \to (8,1)$

ii Horizontal stretch with scale factor 2: $(1,1) \to (2,1)$

Translation $\begin{pmatrix} 6 \\ 0 \end{pmatrix}$: $(2,1) \to (8,1)$

b i Translation $\begin{pmatrix} 3 \\ 0 \end{pmatrix}$: $x^2 \to (x-3)^2$

Horizontal stretch with scale factor 2: $x^2 \to \left(\frac{1}{2}x - 3\right)^2$

ii Horizontal stretch with scale factor 2: $x^2 \to \left(\frac{1}{2}x\right)^2$

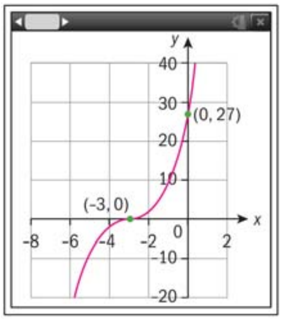
Translation $\begin{pmatrix} 6 \\ 0 \end{pmatrix}$: $\left(\frac{1}{2}x\right)^2 \to \left(\frac{1}{2}(x-6)\right)^2 = \left(\frac{1}{2}x - 3\right)^2$

The sets of transformations are the same because these are the same equation.

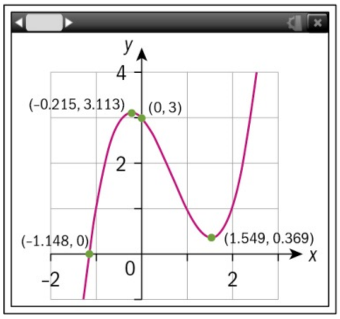
A translation of 3 done before the stretch is equivalent to a translation of 6 done after the stretch.

Exercise 6G

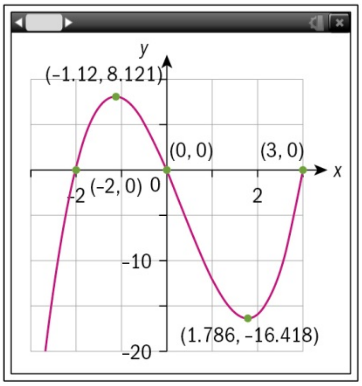
1



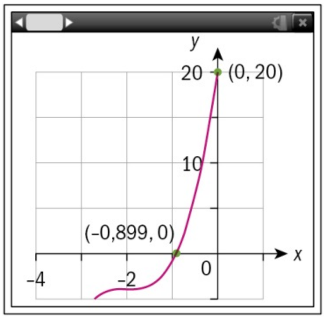
2



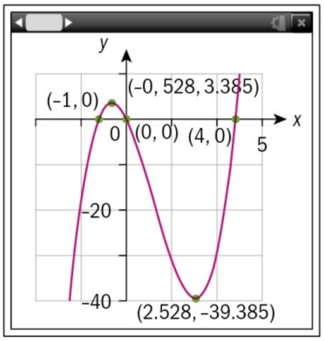
3



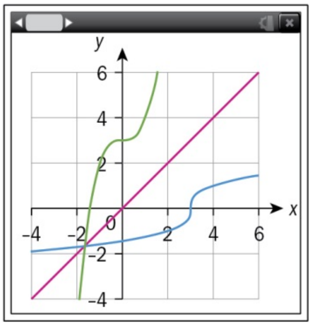
4



5



6 a



By reflecting $y = x^3 + 3$ in the line $y = x$, you get $y = (x - 3)^{\frac{1}{3}}$.

You can find the equation of the inverse algebraically by switching *x* and *y*, and making *y* the subject:

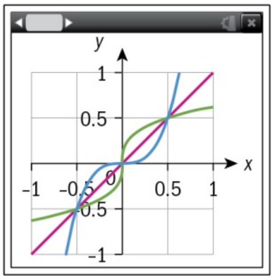
$x = y^3 + 3$

$x - 3 = y^3$

$y = (x - 3)^{\frac{1}{3}}$

So $f^{-1}(x) = (x - 3)^{\frac{1}{3}}$

b



By reflecting $y = 4x^3$ in the line $y = x$, you get $y = \left(\frac{1}{4}x\right)^{\frac{1}{3}}$.

You can find the equation of the inverse algebraically by switching *x* and *y*, and making *y* the subject:

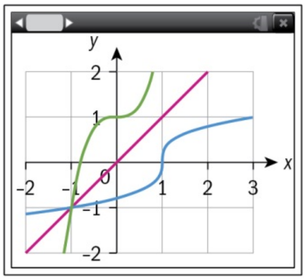
$x = 4y^3$

$\frac{1}{4}x = y^3$

$y = \left(\frac{1}{4}x\right)^{\frac{1}{3}}$

So $f^{-1}(x) = \left(\frac{1}{4}x\right)^{\frac{1}{3}}$

c



By reflecting $y = 2x^3 + 1$ in the line $y = x$, you get $y = \left(\frac{x-1}{2}\right)^{\frac{1}{3}}$

You can find the equation of the inverse algebraically by switching x and y, and making y the subject:

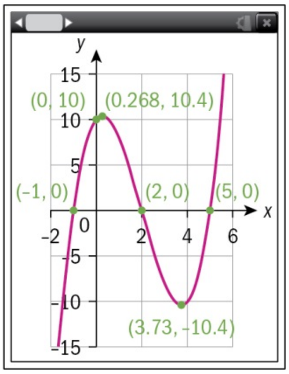
$x = 2y^3 + 1$

$\frac{x-1}{2} = y^3$

$y = \left(\frac{x-1}{2}\right)^{\frac{1}{3}}$

So $f^{-1}(x) = \left(\frac{x-1}{2}\right)^{\frac{1}{3}}$

7



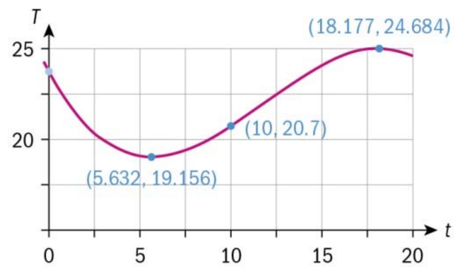
a Using GDC, axes intercepts are (–1, 0), (0, 10), (2, 0) and (5, 0).

b Using GDC, vertices are (0.268, 10.4) and (3.73, –10.4)

c Reflecting $f(x)$ in the y-axis gives the graph of $f(-x)$. This has equation

d $f(-x) = (-x)^3 - 6(-x)^2 + 3(-x) + 10 = -x^3 - 6x^2 - 3x + 10$

8

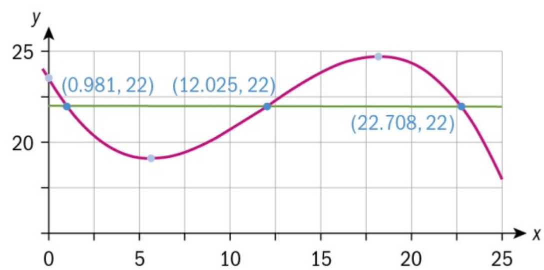


Note that the range of the function is $0 \le t \le 24$. The model is not valid outside this range.

a From GDC, highest temperature is 24.7 °C and lowest temperature is 19.2 °C.

b 05.00 on Tuesday morning is 10 hours after 19.00 on Monday evening. When $t = 10$, then $T = 20.7$ °C.

c

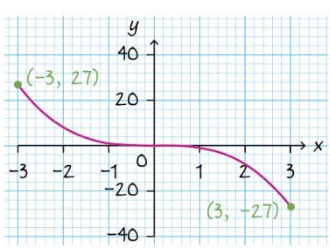


$T > 22$ for $0 \le t < 0.981$ and $12.025 \le t < 22.708$

So air conditioning must be on for $0.981 + (22.708 - 12.025) = 11.7$ hours (1 d.p.)

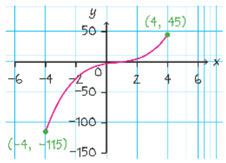
d The function descends rapidly after t = 24, so is not a realistic model for the temperature when t > 24. As such, the model would not be helpful to predict the temperature at 01.00 on Wednesday.

9 **a**



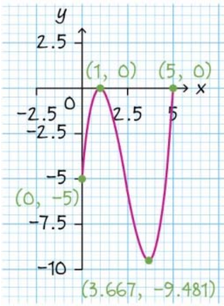
From the GDC or sketch, $-27 \le f(x) \le 27$

b



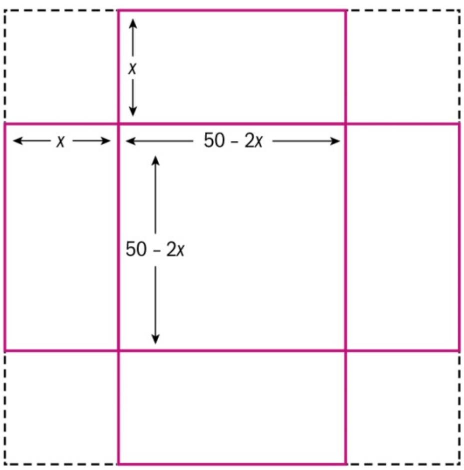
From the GDC or sketch, $-115 \le f(x) \le 45$

c



From the GDC or sketch, $-9.48 \le f(x) \le 0$

10 a



The sketch shows the open box would have:

length = width = $(50 - 2x)$ cm and height = x cm

b Hence, Volume $= x(50-2x)^2$

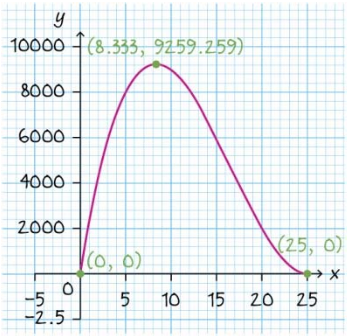
c We must have $x > 0$ in order for the box to have sides to fold up.

Also, we must have $50 - 2x > 0$ in order for the box to have a base.

$50 - 2x > 0 \Rightarrow x < 25$

Hence $0 < x < 25$

d



e From the sketch, the maximum volume occurs at the vertex of the graph.

Max volume = 9259 cm^3 when x = 8.33 cm.

Length = width = 33.33 cm

Height = 8.33 cm

11 Use 'table' function in GDC to generate the following results. In column 2, you should enter the function $0.162t^3 - 3.36t^2 + 18.2t + 1.74$.

t	$T(t)$
1	16.74
2	26.00
3	30.47
4	31.148
5	28.99
6	24.97
7	20.07
8	15.24
9	11.48
10	9.74
11	11.00
12	16.24

a April is month 4. The table shows the max temperature in month 4 is 31.1 °C.

b The lowest max temperature of 9.74 °C occurs in month 10 (October).

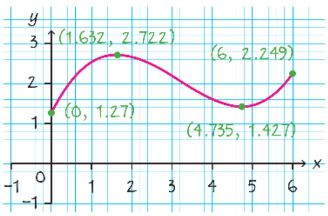
The highest max temperature of 31.1 °C occurs in month 4 (April).

c Mean = $\dfrac{9.74 + 31.148}{2} = 20.44\,°\text{C}$

d A max temperature of 20.07 °C occurs in July (month 7), which is very similar to the average found in part c.

e Month 7 is midway between month 4 and 10. The max temperature in month 7 is roughly equal to the mean max temperature of months 4 and 10. This suggests the temperature falls uniformly between month 4 and month 10.

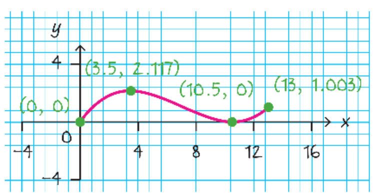
12 a



b From the sketch or GDC, you can see that the range of the function in the domain $0 \le x \le 6$ is $1.27 \le f(x) \le 2.722$.

c The smallest rectangle would have to have length $= 6 - 0 = 6$ mm and height $= 2.722 - 1.27 \approx 1.45$ mm.

13 a



b Roof spans 13 × 200 cm = 26 m.

c Model roof reaches a maximum of 2.117cm above its lowest base.

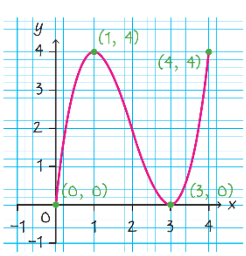
d The function gives the size of the model in centimetres. For the actual roof in metres, a vertical and horizontal scaling of 2 is applied:

$$y = \frac{2}{81}\left(\frac{1}{2}x\right)^3 - \frac{14}{27}\left(\frac{1}{2}x\right)^2 + \frac{49}{18}\left(\frac{1}{2}x\right) = \frac{1}{324}x^3 - \frac{7}{54}x^2 + \frac{49}{36}x$$

where x and y are in metres. A translation of 24 m upwards in the y direction is also required:

$$y = \frac{1}{324}x^3 - \frac{7}{54}x^2 + \frac{49}{36}x + 24$$

14 a



b From the sketch, y-intercept at (0, 0).

c From the sketch or GDC, max at (1, 4) and min at (3, 0).

d Range is $0 \le f(x) \le 4$.

e Let $g(x)$ represent a reflection in the x-axis. Then $g(x) = -f(x)$.

Let $h(x)$ represent vertical stretch by scale factor $\frac{1}{2}$. Then $h(x) = \frac{1}{2}f(x)$.

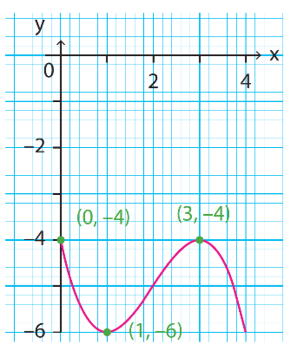
Let $j(x)$ represent translation $\begin{pmatrix} 0 \\ -4 \end{pmatrix}$. Then $j(x) = f(x) - 4$.

f
$$
\begin{aligned}
(j\ \ h\ \ g\ \ f)(x) &= (j\ \ h)(-f(x)) \\
&= j\left(-\frac{1}{2}f(x)\right) \\
&= -\frac{1}{2}f(x) - 4
\end{aligned}
$$

g
$$
\begin{aligned}
y &= -\frac{1}{2}f(x) - 4 \\
&= -\frac{1}{2}\left(x^3 - 6x^2 + 9x\right) - 4 \\
&= -\frac{1}{2}x^3 + 3x^2 - \frac{9}{2}x - 4
\end{aligned}
$$

All of the transformations were in the y-direction, so the domain is unchanged and is $0 \le x \le 4$

h

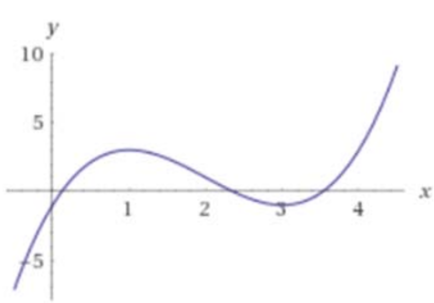


i For the original logo, a positive coefficient of x^3 means that the graph increases to a maximum first, descends to a minimum, and then goes off to infinity as x becomes large.

For the transformed logo, a negative coefficient of x^3 means that the graph decreases to a minimum first, ascends to a maximum, and then goes off to minus infinity as x becomes large.

j The constant term determines the value of the y-intercept.

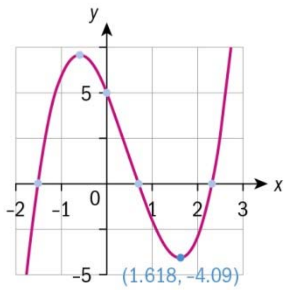
15 a



(x from −0.5 to 4.5)

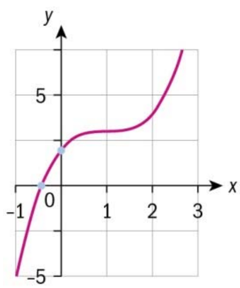
The function has a maximum at (1, 3) and a minimum at (3, -1). Hence, it is invertible for $x \leq 1$, $1 \leq x \leq 3$ or $x \geq 3$ or any subset of these.

b



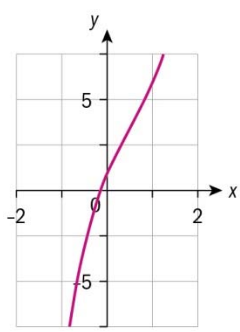
The function has a maximum at (-0.618, 7.090) and a minimum at (1.618, -4.090). Hence, it is invertible for $x \leq -0.618$, $-0.618 \leq x \leq 1.618$ or $x \geq 1.618$ or any subset of these.

c



By plotting $y = x^3 - 3x^2 + 3x + 2$ on your GDC, you can see that the function is one-to-one along its entire domain, so is invertible for $x \in$.

d



By plotting $y = 2x^3 - 3x^2 + 6x + 1$ on your GDC, you can see that the function is one-to-one along its entire domain, so is invertible for $x \in$.

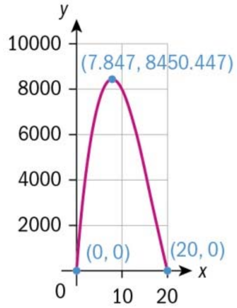
16 Let the square cut from each corner have side length x.

Then the dimensions of the box are $(40 - 2x)$, $(60 - 2x)$, x.

The volume is therefore $V = (40 - 2x)(60 - 2x)x$.

Now $x > 0$ for the box to have sides, and also $40 - 2x > 0 \Rightarrow x < 20$ for the box to have width. Hence the domain of the volume function is $0 < x < 20$

Plotting a graph of V against x for $0 < x < 20$ on your GDC gives

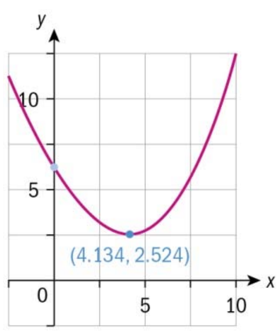


From your GDC, you can see that the max value of this function is $V = 8450 \text{ cm}^3$ when $x = 7.847 \text{ cm}$. Hence, the squares have side 7.85 cm.

Exercise 6H

1 **a** Using cubic regression on GDC, $h(t) = 0.00758t^3 + 0.154t^2 - 1.662t + 6.227$

b



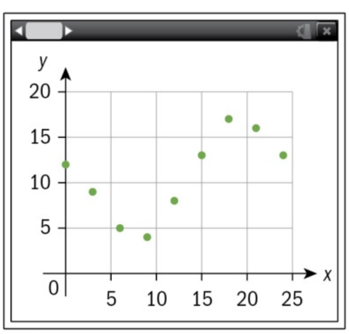
By plotting a graph of the regression function on your GDC, this model suggests the minimum height of the tide is 2.5 m. (Notice that this model is approximate, as the actual height of the tide at $t = 4$ is $h = 2$ m, which is less than the minimum height predicted by this model.)

c Using cubic regression on GDC, $h(t) = 0.245t - 1.936t + 6.355$

d For the cubic curve, $R^2 = 0.955$. For the quadratic curve, $R^2 = 0.951 < 0.955$

Hence, the cubic function is a better model, but only very slightly.

2 **a**



x_1	0	3	6	9	12	15	18	21	24
y_1	12	9	5	4	8	13	17	16	13

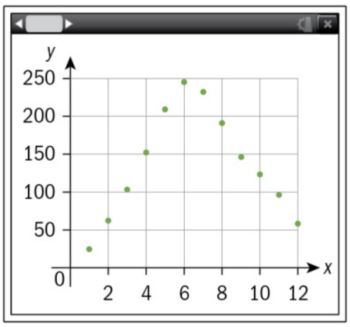
Using the GDC, a best-fit cubic curve to model this data is
$T(t) = -0.00848t^3 + 0.335t^2 - 3.162t + 13.283$

b By observation, the shape of the points in the scatter diagram in part a follows the general shape of a cubic, as it has both a minimum and a maximum point. The coefficient of determination is $R^2 = 0.915$, which is very strong and confirms our observation.

c Using the model to approximate the temperature within the times recorded is interpolation, and the model would be valid for these times.

Using the model to approximate the temperature for the next day is extrapolation, and the model might not be valid for these times.

3 **a**



x_1	1	2	3	4	5	6	7	8	9	10	11	12
y_1	24	62	103	152	209	245	232	191	146	123	96	58

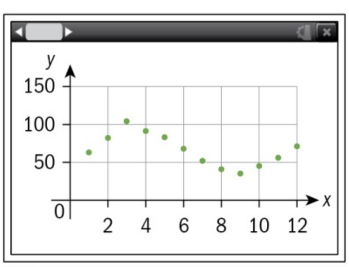
Using the GDC, a best-fit cubic curve to model this data is $y = 0.153x^3 - 9.15x^2 + 98.8x - 87.5$

b The scatterplot shows that the data appears to have only one vertex. This could mean that the data is in the shape of a parabola, in which case a best fit quadratic curve would be appropriate, or it could mean that a portion of a cubic curve might be best.

The coefficient of regression for the quadratic curve is $R^2 = 0.909$, and for the cubic curve is $R^2 = 0.914$. This is marginally better, so we will use a best fit cubic curve.

c We do not know whether the data will continue to behave like this cubic function in the future, so the model is not useful in predicting the number of future cases.

4 **a**



x_1	1	2	3	4	5	6	7	8	9	10	11	12
y_1	63	82	104	91	83	68	52	41	35	45	56	71

Using the GDC, a best-fit cubic curve to model this data is
$y = 0.504x^3 - 9.57x^2 + 46.4x + 27.2$

b By observation, the shape of the points in the scatter diagram in part a follows the general shape of a cubic, as it has both a minimum and a maximum point.

c The coefficient of regression is $R^2 = 0.947$, which is very strong and confirms our observation from b. A cubic model is appropriate.

5 a i $y = -2.71x^2 + 18.7x - 7.40$

ii $R^2 = 0.902$

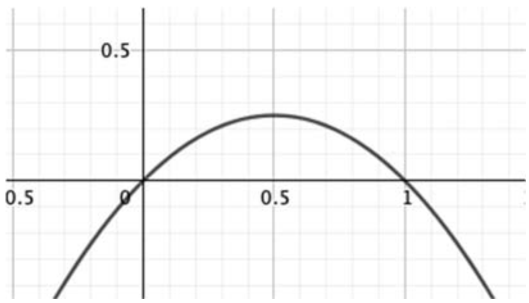
iii $y = 0.500x^3 - 7.96x^2 + 34.5x - 20.0$

iv $R^2 = 0.955$

b The cubic model has a slightly higher coefficient of determination, so would be more appropriate to use when estimating values within $0 \le x \le 6$.

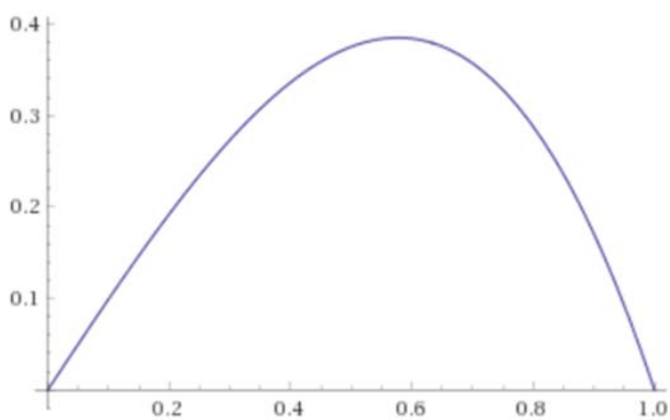
c A particle moving under gravity will always follow a parabolic path, so the quadratic function would be a better model.

6 a i



Maximum is at (0.5, 0.25).

ii



There is a maximum at (0.577, 0.385) so the maximum value between 0 and 1 is 0.385.

7 a A quadratic model might be appropriate as the ball follows the general path of a parabola.

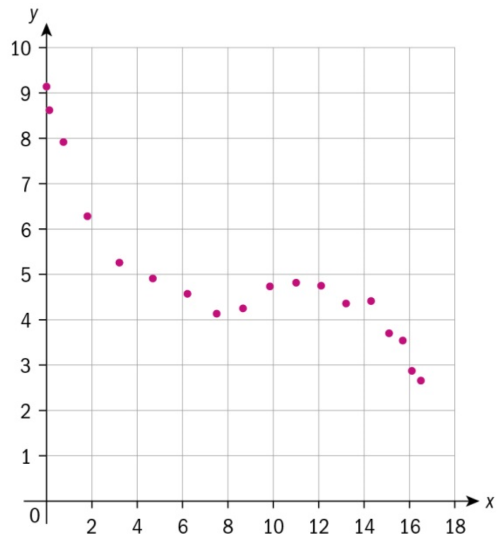
However, it seems that the path is not quite symmetrical about the vertex (which a parabola is), so a quadratic model may not be appropriate.

b A cubic model does not need to be symmetric about a maximum point, so might be a better model.

c The theory does not take into account air resistance. When air resistance is considered, quadratic functions are not wholly appropriate.

8 a

x_1	y_1
0	9.14
0.126	8.62
0.75	7.91
1.8	6.28
3.21	5.26
4.69	4.91
6.22	4.57
7.5	4.13
8.66	4.25
9.85	4.73
11	4.82
12.1	4.75
13.2	4.36
14.3	4.41
15.1	3.7
15.7	3.54
16.1	2.87
16.5	2.66



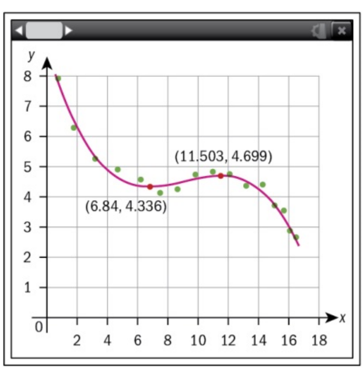
By plotting the points on a scatter diagram, it appears that the F-hole approximately follows the shape of a cubic curve, as it is both concave up and down and because it is approximately symmetric about a point

b Using GDC, $y = -0.00716x^3 + 0.197x^2 - 1.69x + 8.97$

This is valid within the domain of x-values given, so $0 \le x \le 16.5$.

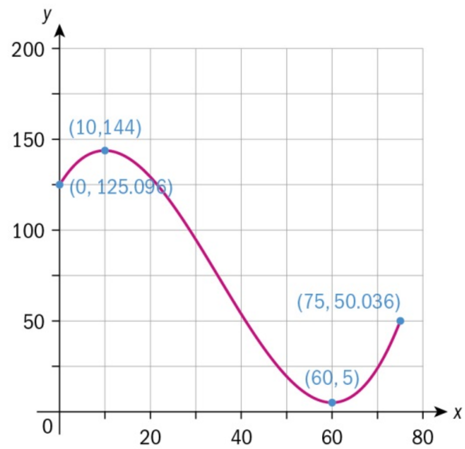
c The coefficient of determination is $R^2 = 0.991$, which is very strong, so this model is appropriate.

d



By examining the best fit cubic curve, it has a minimum at (6.80, 4.34) and a maximum at (11.6, 4.73).

9

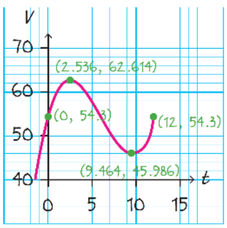


a Using GDC, max height is 144 m

b Using GDC, min height is 5 m.

c Roller coaster descends 144 – 5 = 139 m, so this is the greatest vertical descent (beating the previous one by 9 m).

10 a

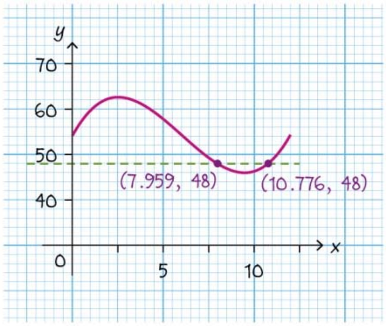


Plot the graph on your GDC and then transfer it to paper. There are 12 months in a year, so the domain should be $0 \le t \le 12$.

b From GDC, there is a maximum of 62.6 km^3 when t = 2.54 months.

c From GDC, there is a maximum of 46.0 km^3 when t = 9.46 months.

d



By drawing the line y = 48 on your sketch (or GDC), you can see the water falls below 48 km^3 between t = 7.96 and t = 10.78.

t = 7.96 is in the 8th month, which is August. t = 10.78 is in the 11th month, which is November.

So the water warning is active in parts of August, September, October and November.

Volume of a cone $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi r^2 \times \pi r = 35000\,\text{cm}^3$ (2 significant figures)

Exercise 6I

1 **a** If resistance varies directly with the velocity, then $R = kv$ for constant k.

$v = 2, R = 4.2 \Rightarrow k = \frac{R}{v} = \frac{4.2}{2} = 2.1$

So $R = 2.1v$

If resistance varies with the square of the velocity, then $R = av^2$

$v = 2, R = 4.2 \Rightarrow a = \frac{R}{v^2} = \frac{4.2}{2^2} = 1.05$

So $R = 1.05v^2$

b Linear model: $R(3.2) = 2.1 \times 3.2 = 6.72$, $R(4.0) = 2.1 \times 4.0 = 8.4$

Quadratic model: $R(3.2) = 1.05 \times 3.2^2 = 10.752$, $R(4.0) = 1.05 \times 4.0^2 = 16.8$

c We calculate the residuals for the linear model using this table:

v	3.2	4.0
R_{true}	7.1	11.1
R_{model}	6.72	8.4
$R_{\text{true}} - R_{\text{model}}$	0.38	2.7
$(R_{\text{true}} - R_{\text{model}})^2$	0.144	7.29

Sum of squares = $0.144 + 7.29 = 7.43$

We calculate the residuals for the quadratic model using this table:

v	3.2	4.0
R_{true}	7.1	11.1
R_{model}	10.752	16.8
$R_{\text{true}} - R_{\text{model}}$	-3.652	-5.7
$(R_{\text{true}} - R_{\text{model}})^2$	13.34	32.49

Sum of squares = $13.34 + 32.49 = 45.83$

The linear model is much more likely as the sum of squares of residuals is much smaller than that for the quadratic model.

d Using GDC, $R = 1.34v^{1.51}$

e Using GDC, the residuals for the power regression model are -0.384, 0.660 and -0.230.

Sum of squares of residuals is 0.636. This is much smaller than the sum of squares of the linear model, so the relationship between v and R is likely to be a power function.

2 **a** $d = kt^2$

$t = 2, d = 9 \;\Rightarrow\; 9 = 4k \;\Rightarrow\; k = \dfrac{9}{4}$

$d = \dfrac{9}{4}t^2$

b $d(5) = \dfrac{9}{4} \times 5^2 = 56.25$ m

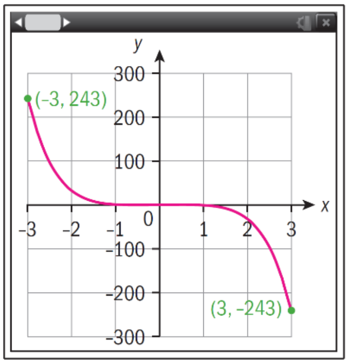
c $26.01 = \dfrac{9}{4}t^2 \Rightarrow t = \sqrt{\dfrac{4}{9} \times 26.01} = 3.4$ s

3 $m = kr^3$

$m = 113.1,\; r = 3 \;\Rightarrow\; 113.1 = k \times 27 \;\Rightarrow\; k = 4.188\ldots$

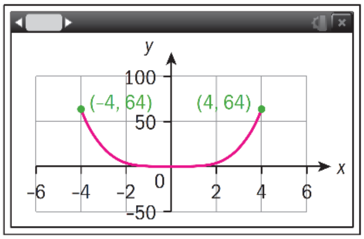
$m(5) = 4.1888\ldots \times 5^3 = 523.6 \approx 524$ g

4 a



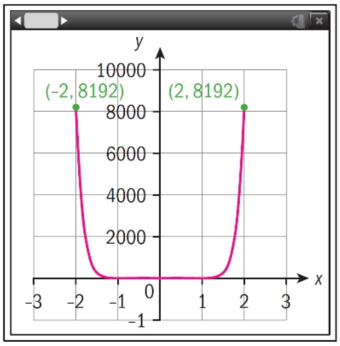
From your GDC, the range of the function is $-243 \le y \le 243$.

b



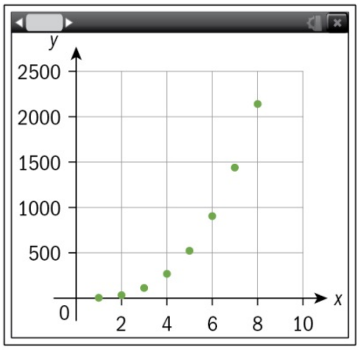
From your GDC, the range of the function is $0 \le y \le 64$.

c



From your GDC, the range of the function is $0 \le y \le 8192$.

5 a

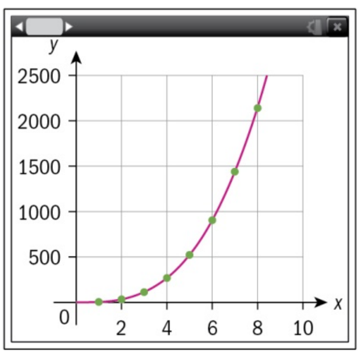


x_1	1	2	3	4	5	6	7	8
y_1	4.19	33.5	113	268	524	905	1440	2140

b By using the power regression function on your GDC, $V = 4.19r^{3.00}$.

c The coefficient of determination $R^2 = 1.000$ so the power model fits this data almost perfectly.

d



x_1	1	2	3	4	5	6	7	8
y_1	4.19	33.5	113	268	524	905	1440	2140

As stated in part c, this power function is almost a perfect fit.

e $V(10) = 4.19(10)^{3.00} = 4190\ \text{cm}^3$

f $V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi \times 10^3 \approx 4188.79\ldots\ \text{cm}^3$. This is very close to the answer from part **e**.

6 a $W = kp^4$

$p = 6,\ W = 55\,000 \Rightarrow 55\,000 = k \times 6^4 \Rightarrow k = \frac{6875}{162}$

$W = \frac{6875}{162}p^4$

b $W(8) = \frac{6875}{162} \times 8^4 = 173\,827\ \text{kg}$

c $p = \sqrt[4]{\frac{162}{6875}W}$

$$p(200\ 000) = \sqrt[4]{\frac{162}{6875}(200\ 000)} = 8.29 \text{ m}$$

7 **a** $f^{-1}(x) = -\sqrt{x},\ x \geq 0$

b Let $y = -x^3 \Rightarrow x = \sqrt[3]{-y}$ so $f^{-1}(x) = \sqrt[3]{-x}$

c $y = 2x^5 \Rightarrow x = \sqrt[5]{\frac{y}{2}}$ so $f^{-1}(x) = \sqrt[5]{\frac{x}{2}}$

Exercise 6J

1 $V = \frac{k}{p}$

$V = 180, p = 20 \Rightarrow 180 = \frac{k}{20} \Rightarrow k = 3600$

$V = \frac{3600}{p}$

$V(90) = \frac{3600}{90} = 40$ Pa

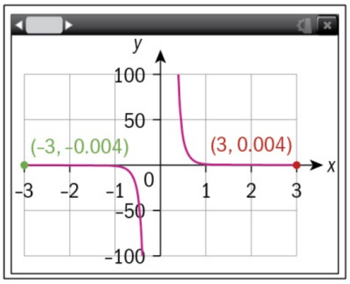
2 $c = \frac{k}{n}$

$10 = \frac{k}{16} \Rightarrow k = 160$

$c = \frac{160}{n}$

$c(20) = \frac{160}{20} = 8$

3 **a**

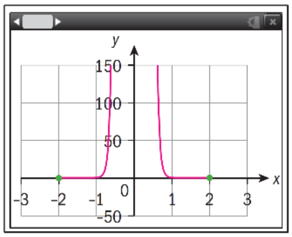


This has a vertical asymptote at $x = 0$, and $\frac{1}{x^5} \to \pm\infty$ as $x \to \pm 0$

$f(-3) = \frac{1}{(-3)^5} = -\frac{1}{243} = -0.004115$ and $f(3) = 0.004115$

So the range is $f(x) \leq -0.004115$ and $f(x) \geq 0.004115$

b

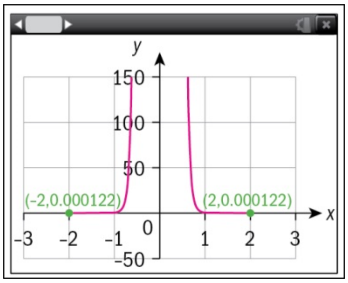


This has a vertical asymptote at $x = 0$, and $\frac{4}{x^4} \to +\infty$ as $x \to \pm 0$

$g(-4) = g(4) = 0.016$

So the range is $g(x) \geq 0.016$

c



This has a vertical asymptote at $x = 0$, and $\frac{1}{2x^{12}} \to +\infty$ as $x \to \pm 0$

$h(-2) = h(2) = \frac{1}{8192}$

So the range is $f(x) \geq \frac{1}{8192}$.

4 $d = \frac{k}{r^3}$

$d = 500, r = 0.1 \Rightarrow 500 = \frac{k}{0.1^3} \Rightarrow k = 0.5$

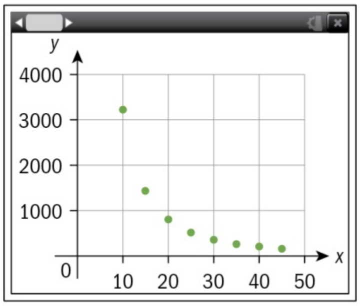
$d = \frac{1}{2r^3}$

a $d(5) = \frac{1}{2 \times (0.05)^3} = 4000 \text{ kg/m}^3$

b $r = \frac{1}{(2d)^{\frac{1}{3}}}$

$r(100) = \frac{1}{(2 \times 100)^{\frac{1}{3}}} = 0.171 \text{ m} = 17.1 \text{ cm}$

5 **a**



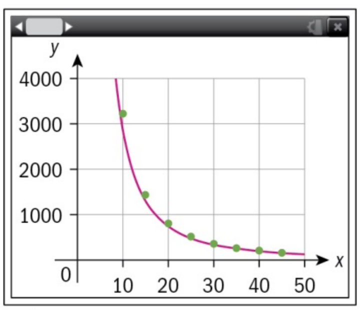
x_1	10	15	20	25	30	35	40	45
y_1	3232	1434	805	513	535	259	201	159

b The data looks as though it follows the general shape of an inverse variation curve.

c $I = \dfrac{329000}{x^{2.01}}$

d $R^2 = 1.000$, which is a very strong coefficient of determination, so the inverse variation function is an appropriate model.

e

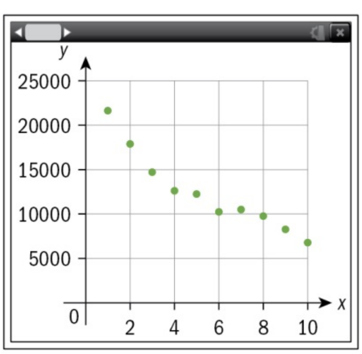


x_1	10	15	20	25	30	35	40	45
y_1	3232	1434	805	513	535	259	201	159

This is a close fit.

f $I(50) = \dfrac{328000}{(50)^{2.01}} = 127$ lux

6 **a**



x_1	1	2	3	4	5	6	7	8	9	10
y_1	21641	17890	14709	12613	12239	10223	10501	9744	8257	6772

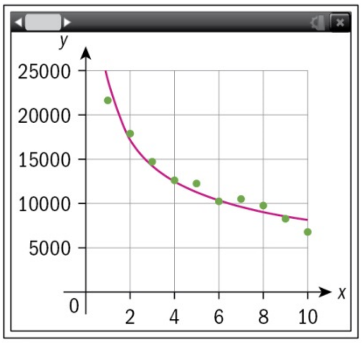
b The price of a car is not quite linear since the gradient becomes less steep over time. If it were linear with a suitable gradient, the value of the car would fall much too quickly.

c The depreciation of the car is quite a lot in the early years, but levels off as the car gets older. An inverse variation function could model this.

d $P = 23688t^{-0.46253}$ where P is the price in euros and t is the age in years

e $R^2 = 0.958$, which is a very strong coefficient of determination, so the inverse variation function is an appropriate model.

f



x_1	1	2	3	4	5	6	7	8	9	10
y_1	21641	17890	14709	12613	12239	10223	10501	9744	8257	6772

This model fits the general shape of the data well. There are a similar number of points above and below the curve.

g $4000 = 23\,688t^{-0.46253}$

Solving using GDC or logs $\Rightarrow t = 46.8$ years

7 a A quick sketch shows it is symmetrical about the y-axis, so function is invertible for either $x > 0$ or $x < 0$.

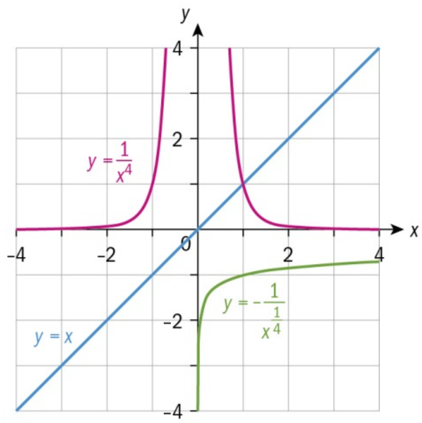
b This is one-to-one on all real numbers, so invertible for $x \in$

c A quick sketch shows it is symmetrical about the y-axis, so function invertible for either $x > 0$ or $x < 0$.

8 a Let $y = \frac{1}{x^4} \Rightarrow x = \pm\frac{1}{y^{\frac{1}{4}}}$

Since domain of f is $x < 0 \Rightarrow f^{-1}(x) = -\frac{1}{x^{\frac{1}{4}}}$

You can check you have the correct sign of the inverse function by plotting it on your GDC. The inverse function should be a reflection in $y = x$ of the correct branch of the original function.



b Let $y = \frac{2}{x^3} \Rightarrow x = \sqrt[3]{\frac{2}{y}} \Rightarrow f^{-1}(x) = \sqrt[3]{\frac{2}{x}}$

c $y = \frac{1}{x^2} \Rightarrow x = \pm\frac{1}{\sqrt{y}}$

Since domain of f is $x > 0 \Rightarrow f^{-1}(x) = \frac{1}{\sqrt{x}}$

9 a $\frac{1}{x} \to \frac{1}{x-1}$ translation by vector $\begin{pmatrix}1\\0\end{pmatrix}$

$\frac{1}{x-1} \to \frac{2}{x-1}$ stretch in y-direction with scale factor 2

$\frac{2}{x-1} \to -\frac{2}{x-1}$ reflection in x-axis

$-\frac{2}{x-1} \to 3-\frac{2}{x-1}$ translation by vector $\begin{pmatrix}0\\3\end{pmatrix}$

b $\frac{1}{x}$ has a vertical asymptote at $x = 0$ as the function is not defined here.

$\frac{1}{x}$ has an horizontal asymptote at $y = 0$ as $\frac{1}{x} \to 0$ as $x \to \pm\infty$

$3-\frac{2}{x-1}$ has a vertical asymptote at $x = 1$ as the function is not defined here.

$3-\frac{2}{x-1}$ has an horizontal asymptote at $y = 3$ as $\frac{2}{x-1} \to 0$ as $x \to \pm\infty$

c The same transformations which map f onto g also map the asymptotes of f onto the asymptotes of g.

10 $1-\frac{2}{x-2} \to -\frac{2}{x-2}$ translation by vector $\begin{pmatrix}0\\-1\end{pmatrix}$

$-\frac{2}{x-2} \to \frac{2}{x-2}$ reflection in x-axis

$\frac{2}{x-2} \to \frac{1}{x-2}$ stretch in y-direction with scale factor $\frac{1}{2}$

$$\frac{1}{x-2} \to \frac{1}{x} \text{ translation by vector } \begin{pmatrix} -2 \\ 0 \end{pmatrix}$$

11 a i The shape of the demand curve resembles an inverse power law. Thus, $n < 0$.

ii As p gets large, this function suggests that a small percentage of the market might still buy the product. This is unlikely to be realistic when the price is unreasonably high.

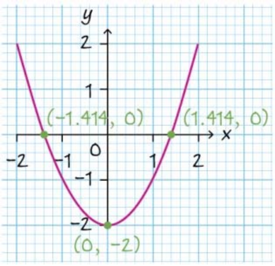
b $N = kp^n$

$(1, 57) \Rightarrow 57 = k$

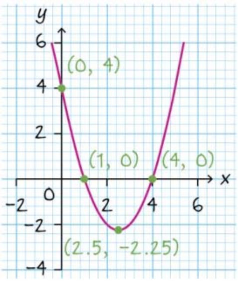
$(4, 13) \Rightarrow 13 = 57 \times 4^n \Rightarrow n = -1.066$

Chapter review

1 a



b



2 a Let h be the height of the picture.

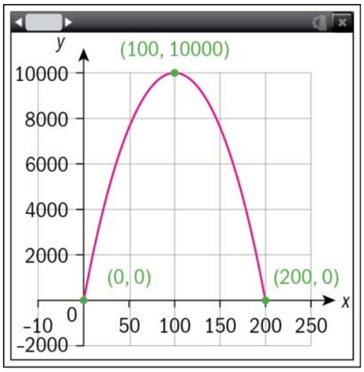
$$2x + 2h = 400$$
$$h = \frac{400 - 2x}{2}$$
$$h = 200 - x$$

b $A = xh$

$= x(200 - x)$

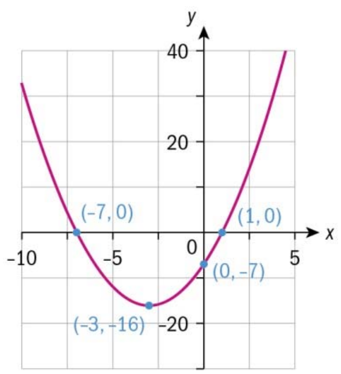
$= 200x - x^2$

c



d Using GDC, x-intercepts are (0,0) and (200,0), which represent the limiting values of x for which the picture can be a rectangle.

3



From GDC:

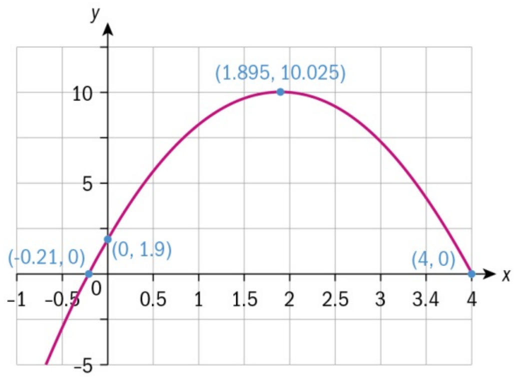
a y-intercept (0, –7)

b x-intercepts (–7, 0) and (1, 0)

c Axis of symmetry passes through the vertex and has equation $x = -3$

d Vertex is (–3, –16)

4



a From GDC, y-intercept (0, 1.9) means the stone is thrown from a height of 1.9 m.

b The max height of the stone occurs at the vertex of the graph, which is 10.0 m.

c The stone lands back on the ground when the graph intercepts the positive x-axis, at 4 seconds.

5 a i The equation is already written in intercept form, so x-intercepts are $x = 2$ and $x = 4$.

ii the vertex is the average of the two roots of the function, $x = 3$.

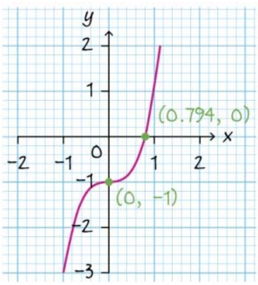
iii Vertex is (3, –3)

b i The equation is already written in intercept form, so x-intercepts are $x = -1$ and $x = 5$.

ii Plotting the graph on your GDC shows the axis of symmetry, which passes through the vertex, is $x = 2$.

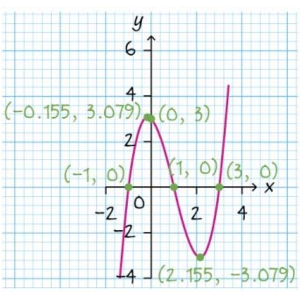
iii Vertex is (2, –36)

6 a



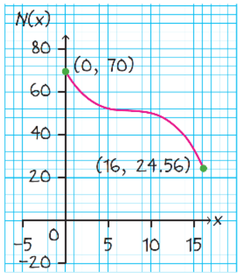
Start by plotting $y = 2x^3 - 1$ on your GDC, then transfer to paper. Make sure to label the coordinates of the intercepts.

b



The curve is written in intercept form, so you can see the three x-intercepts are $x = 1, -1, 3$. Mark these on your sketch. By plotting the graph on your GDC, you can also see the coordinates of the vertices, which you should mark on your sketch.

7 a



b Either by using GDC to find the N-coordinate when $x = 5$, or by calculating $N(5) = -0.04(5)^3 + 0.9(5)^2 - 7(5) + 70 = 52.5$ gives 52.5 lilies.

c Using either method from part b gives 46.5 lilies after 12 years.

d From the graph, you can see that the maximum number of lilies is 70, and this occurs when $x = 0$ which represents the year 2004.

e From the graph, the minimum number of lilies occurs when $x = 16$, and is 25 lilies. The year is 2020.

f From the graph, there are 60 lilies when $x = 1.82$ years. This is during the year 2005.

8 **a** From the equation distance = speed × time, we know that

$$m = kt$$
$$m = 100,\ t = 1.25 \Rightarrow 100 = 1.25k \Rightarrow k = 80$$
$$m = 80t$$

b $m(2) = 80 \times 2 = 160$ miles

c $300 = 80t \Rightarrow t = \dfrac{300}{80} = 3.75$ hours

9 EITHER: You can plot both curves on your GDC and find the coordinates of the intersection point,

OR calculate the intersection point algebraically:

Line has equation $y = \dfrac{1}{2}x + \dfrac{3}{2}$

Substituting this into the equation for $f(x)$ gives

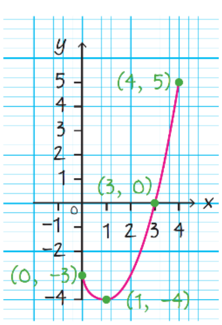
$$\frac{1}{2}x + \frac{3}{2} = x^2 + 2x - 5$$
$$0 = x^2 + \frac{3}{2}x - \frac{13}{2}$$
$$0 = 2x^2 + 3x - 13$$

Solving using GDC $\Rightarrow x = -3.41$ and $x = 1.91$

Substituting into $y = \dfrac{1}{2}x + \dfrac{3}{2}$ gives $y = -0.204$ and $y = 2.45$

Intersect at (–3.41, –0.204) and (1.91, 2.45)

10



The sketch shows that the minimum value of $f(x)$ on the domain $0 \le x \le 4$ occurs at the vertex, where $y = -4$. The maximum value of $f(x)$ on $0 \le x \le 4$ occurs at $x = 4$, and is $f(x) = 5$. Hence, the range is -4 ≤ y ≤ 5.

11 a $a = \dfrac{k}{\sqrt{t}}$

$a = 6.2,\ t = 5.76 \Rightarrow k = 6.2 \times \sqrt{5.76} = 14.88$

$a = \dfrac{14.88}{\sqrt{t}}$

b $0.5 = \dfrac{14.88}{\sqrt{t}}$

$t = \left(\dfrac{14.88}{0.5}\right)^2 = 886$ seconds

12 a The vertex occurs at the midpoint of the x-intercepts

$h = \dfrac{0+6}{2} = 3$

b Let the curve have equation $y = a(x-p)(x-q)$ where p and q are the known roots. Then $y = ax(x-6)$. For the vertex (3,8):

$$8 = 3a(3-6)$$
$$= 3a(-3)$$
$$a = -\frac{8}{9}$$

Hence,

$$y = -\frac{8}{9}x(x-6)$$
$$= -\frac{8}{9}x^2 + \frac{16}{3}x$$

13 a $(2,32) \Rightarrow 32 = k \times 2^n$ (1)

$(4,2) \Rightarrow 2 = k \times 4^n$ (2)

$(1) \div (2) \Rightarrow 16 = \dfrac{2^n}{4^n} = \dfrac{2^n}{2^{2n}} = 2^{-n}$

$16 = 2^4 \Rightarrow 2^4 = 2^{-n}$

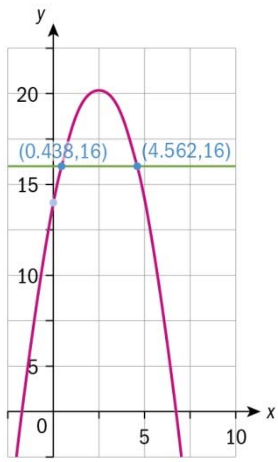
$\Rightarrow n = -4$

Substitute in (2) $\Rightarrow k = 2 \times 4^{-n} = 2 \times 4^4 = 512$

b Using power regression on GDC gives the same results as part (a).

14 $A = (x+2)(7-x)$

Plot $A = (x+2)(7-x)$ and the line $A = 16$ on your GDC:



The area is equal to 16 when $x = 0.438$ and $x = 4.562$

So area greater than 16 when $0.438 < x < 4.562$

15 a $60 = AB + BC + CD$

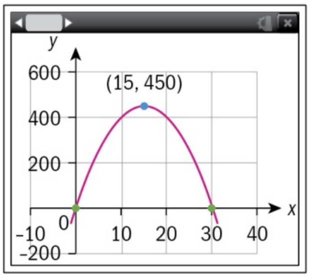
$AB = DC = x$

So $60 = 2x + BC$

$BC = 60 - 2x$

b $A = x(60 - 2x) = 60x - 2x^2$

c



By plotting $y = 60x - 2x^2$ on your GDC, you can see that the max value of the area is $A = 450$ when $x = 15$ m.

16 a The initial height occurs when $t = 0$

$h(0) = 12$ so the ball is released from a height of 12 m.

b $h(1) = -(1)^2 + 6(1) + 12 = -1 + 6 + 12 = 17$ m

c i $17 = -t^2 + 6t + 12$

ii $17 = -t^2 + 6t + 12$

$t^2 - 6t + 5 = 0$

$(t - 5)(t - 1) = 0$

$t = 5$ or $t = 1$

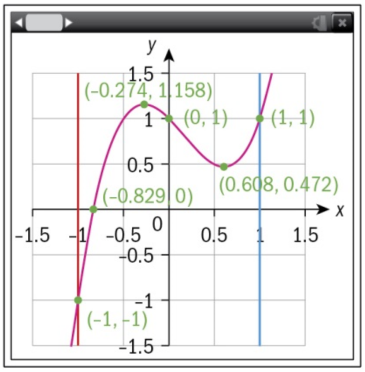
d i Reaches maximum height at the vertex.

Curve is symmetric about the vertex, so vertex occurs at the midpoint between the two t-values at which height is 17 (found in part c ii).

$$t = \frac{1+5}{2} = 3 \text{ s}$$

ii $h(3) = 21$, so max height is 21 m

17



Plot the graph of $y = 2x^3 - x^2 - x + 1$ using your GDC. From the graph, you can see that the range is $-1 \le y \le 1.158$.

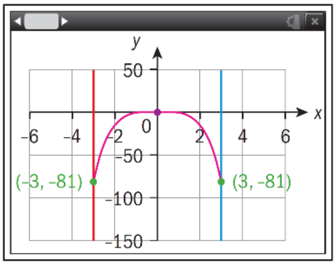
18 a $P(t) = 0.03t^3 + 0.18t^2 - 7.9t + 63$. Values given by the function are

Month	1	2	3	4	5	6	7	8	9	10	11	12
P	55.3	48.2	41.7	36.2	31.8	28.6	26.8	26.7	28.3	32.0	37.8	46.0

July is month 7, so table gives $P(7) = 26.8$ mm (3 s.f.)

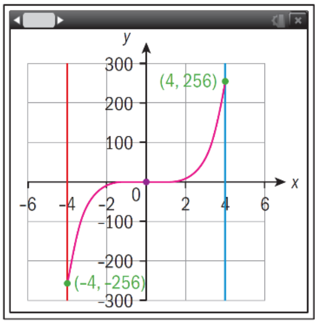
b The least precipitation is in month 8, August and the most in month 1, January

19 a



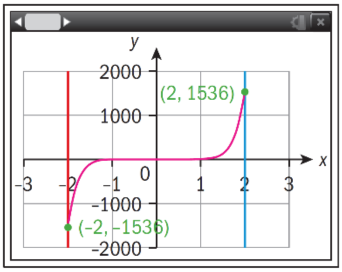
From your GDC, you can see that the range is $-81 \le f(x) \le 0$

b



From your GDC, you can see that the range is $-256 \le g(x) \le 256$.

c



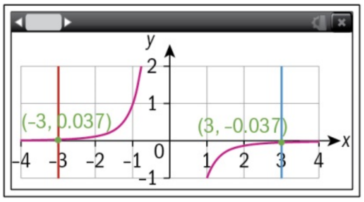
From your GDC, you can see that the range is $-1536 \le h(x) \le 1536$.

20 a By using cubic regression on GDC, $y = -0.00489t^3 + 0.132t^2 - 0.759t + 29.8$

b Using this model to predict future electricity prices would be extrapolation, so the model may not apply outside the given domain. It predicts that the price will fall after 2020 but this cannot be relied on without further information.

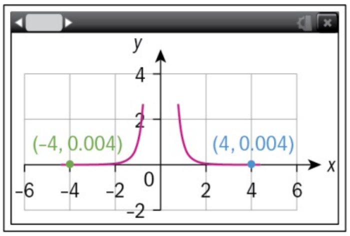
c For $t = 15$ the model suggests electricity will cost about 31.61 Euros.

21 a



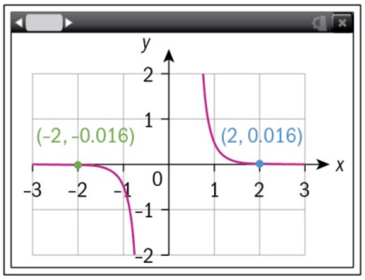
$y \to \pm\infty$ as $x \to 0$ so range is $-\infty < y \le -0.037$ and $0.037 \le y < \infty$

b



$y \to +\infty$ as $x \to \pm 0$ so range is $0.004 \le y < \infty$

c



$y \to \pm\infty$ as $x \to \pm 0$ so range is $-\infty < y \leq -0.016$ and $0.016 \leq y < \infty$

Exam-style questions

22 a $1.7 = 1.7 + 50t - 5t^2 \Rightarrow t = 0$ or $10 \Rightarrow$ rocket returns at $t = 10\,s$. (2)

b By symmetry, maximum must be halfway between 0 and 10.

So reaches max height at $t = 5\,s$ (2)

c $s(5) = 1.7 + 50 \times 5 - 5 \times 5^2 = 126.7\,\text{m}$ (2)

d The displacement-time graph for Paul's rocket is a translation of Peter's graph by 0.1 m in the y-direction, so for Paul's rocket (1)

a 10 s (1)

b 5 s (1)

c 126.8 m (1)

23 a $D = b^2 - 4ac = 4(k^2 - 36)$ (1)

i For one repeated root, $D = 0 \Rightarrow k = 6$ or $k = -6$ (2)

ii For no real roots, $D < 0 \Rightarrow -6 < k < 6$ (2)

iii For two distinct roots, $D > 0 \Rightarrow k < -6$ or $k > 6$ (3)

b $D = b^2 - 4ac = 4(k^2 + 36)$, which is always positive, so there are always 2 roots. (3)

24 a $(3, 7)$ (2)

b Restrict the domain to include only values to the right of the vertex $(3, 7)$. Then the curve is one-to-one and includes the point $(4, 8)$ (1)

Domain $x \geq 3$; Range $y \geq 7$ (2)

c $f : x \to y = (x - 3)^2 + 7$, inverse given by

$x = (y - 3)^2 + 7 \Rightarrow y - 3 = \sqrt{x - 7} \Rightarrow f^{-1}(x) = 3 + \sqrt{x - 7}$ (3)

Domain $x \geq 7$; Range $y \geq 3$ (2)

25 a Using GDC, minimum is $(50, 12.5)$ (3)

b Using GDC, maximum is $(20, 26)$ (3)

c Solving $h(x) > 50$ (by graph) gives $73.4 \le x \le 80$ (2)

26 a $\sqrt{5} \times 10^{\frac{1}{2}} = 7.07$ s (3 s.f.) (2)

b $8 = \sqrt{5}d^{\frac{1}{2}} \Rightarrow d = 12.8$ m (2)

c $d = \sqrt{5}d^{\frac{1}{2}} \Rightarrow d^{\frac{1}{2}} = \sqrt{5} \Rightarrow d = 5$ m (2)

27 $xy = x(24 - x) = 143$ (1)

$\Rightarrow x^2 - 24x + 143 = 0$ (1)

$\Rightarrow x = 11$ or 13 (1)

$x = 11 \Rightarrow y = 13$ or $x = 13 \Rightarrow y = 11$ (1)

So the possible passwords are 1113 or 1311 (2)

28 a $g(x) = x^2 + k$, $g(3) = 4 \Rightarrow 9 + k = 4 \Rightarrow k = -5$ (2)

b $h(x) = (x - l)^2$, $h(3) = 4 \Rightarrow 3 - l = \pm 2 \Rightarrow l = 5$ or 1 (3)

c $m(x) = (x - r)^2 + s$, which has a minimum at (r, s) so $r = 3,\ s = 4$. (3)

d $n(x) = a(x - b)^2 + c$, for this quadratic to have a maximum we require $a < 0$ and $b = 3,\ c = 4$. (4)

29 a Let the quadratic be $ax^2 + bx + c$.

Solving $a + b + c = 10$, $4a + 2b + c = 27$ and $16a + 4b + c = 115$ (1)

(OR by using quadratic regression on GDC)

gives $a = 9,\ b = -10,\ c = 11$ (3)

b $9 \times 5^2 - 10 \times 5 + 11 = 186 \ne 198$. (1)

So point does not lie on the quadratic. (1)

c Let the cubic be $px^3 + qx^2 + rx + s$.

Solving $p + q + r + s = 10$, $8p + 4q + 2r + s = 27$,

$64p + 16q + 4r + s = 115$ and $125p + 25q + 5r + s = 198$ (1)

(OR by using cubic regression on GDC)

gives $p = 1,\ q = 2,\ r = 4,\ s = 3$ (4)

30 Place the origin at the centre of the bridge; then there is a minimum at the point $(0, 2)$ (2)

The equation of the cable will be of the form $y(x) = ax^2 + 2$ (2)

Let the length of the bridge be $2l$. Then $y(l) = al^2 + 2 = 10 \Rightarrow al^2 = 8$ (2)

The height of the cable a quarter of the way along the bridge is given by

$$y\left(\frac{l}{2}\right) = a\left(\frac{l}{2}\right)^2 + 2 \quad (1)$$

$$= \frac{al^2}{4} + 2$$

$$= \frac{8}{4} + 2 = 4 \text{ m} \quad (1)$$

7 Modelling rates of change: exponential and logarithmic functions

Skills check

1 **a** $\left(x^2\right)^3 = x^{2\times3} = x^6$

b $\left(\dfrac{a}{b}\right)^{-1} = \dfrac{a^{-1}}{b^{-1}} = \dfrac{b}{a}$

c $x^{-2}\sqrt{x} = x^{-2}x^{\frac{1}{2}} = x^{-\frac{3}{2}} = \dfrac{1}{\sqrt{x^3}}$

d $\dfrac{\sqrt{x}}{\sqrt[3]{x}} = x^{\frac{1}{2}}x^{-\frac{1}{3}} = x^{\frac{1}{6}} = \sqrt[6]{x}$

e $\left(\sqrt{x}\right)^5 = \left(x^{\frac{1}{2}}\right)^5 = x^{\frac{5}{2}} = \sqrt{x^5}$

2 **a** $0.03 \times 24 = 0.72$

b $0.15 \times 72 = 10.8$

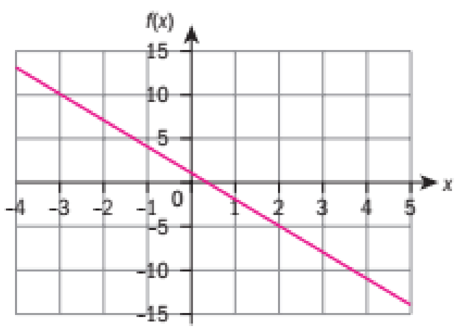
c $0.28 \times 150 = 42$

3 **a**

x	-1	0	1	2
y	4	1	-2	-5

b $f(12) = 1 - (3 \times 12) = -35$

c



4 **a** $\sum_{i=1}^{10}(2i+1) = 3+5+7+9+11+13+15+17+19+21 = 120$

b $\sum_{i=5}^{10}(2i+1) = 11+13+15+17+19+21 = 96$

c $\sum_{i=1}^{4} i^3 = 1+8+27+64 = 100$

Exercise 7A

1 a $r=\frac{10}{5}=2$, $u_7=5\times2^6=320$, $u_n=5\times2^{n-1}$

b $r=\frac{15}{-3}=-5, u_8=-3\times(-5)^7=234375$, $u_n=-3\times(-5)^{n-1}$

c $r=\sqrt{\frac{6}{2}}=\sqrt{3}$, $u_6=\sqrt{2}\times\sqrt{3}^5=\sqrt{2}\times9\times\sqrt{3}=9\sqrt{6}\approx22.0$, $u_n=\sqrt{2}\times\sqrt{3}^{n-1}$

d $r=\frac{1}{3/2}=\frac{2}{3}$, $u_4=\frac{3}{2}\times\left(\frac{2}{3}\right)^3=\frac{4}{9}$, $u_n=\frac{3}{2}\left(\frac{2}{3}\right)^{n-1}=\left(\frac{2}{3}\right)^{n-2}$

e $r=\frac{20}{2}=10$, $u_{10}=2\times(10)^9=2\times10^9$, $u_n=2\times10^{n-1}$

2 a $u_4=24=3\ r^3\Rightarrow r=\sqrt[3]{8}=2$. Hence $u_n=3\times2^{n-1}$

b $u_6=32r^5=243\Rightarrow r=\sqrt[5]{\frac{243}{32}}=1.5$. Hence $u_n=32\times1.5^{n-1}$

c $u_5=81=1r^4\Rightarrow r=\sqrt[4]{81}=3$. Hence $u_n=3^{n-1}$

d $u_4=\frac{1}{2}r^3=\frac{4}{27}\Rightarrow r=\sqrt[3]{\frac{8}{27}}=\frac{2}{3}$. Hence $u_n=\frac{1}{2}\left(\frac{2}{3}\right)^{n-1}$

e $u_7=-1458=-2r^6\Rightarrow r=\pm\sqrt[6]{729}=\pm3$. To give positive and negative terms, $r=-3$. Hence $u_n=-2(-3)^{n-1}$

f $u_3=13.5=u_1r^2$ and $u_6=364.5=u_1r^5$ hence $\frac{u_6}{u_3}=r^3=27\Rightarrow r=3$. Then $u_3=13.5=u_1\times3^2\Rightarrow u_1=\frac{13.5}{9}=1.5$ which gives $u_n=1.5(3)^{n-1}$.

3 a $\frac{u_2}{u_1}=\frac{10}{2}=5=\frac{u_3}{u_2}$ hence, it's a geometric sequence.

b $u_7=2\times5^6=31250$

c $u_{30}=2\times5^{29}=3.73\times10^{20}$. This is obviously not a reasonable answer. The total population of the world is less than 10^{10} people.

Exercise 7B

1 a As the sequence is geometric, then $P_{2018}=P_{2016}r^2\Rightarrow r=\sqrt{\frac{264\ 500}{200\ 000}}=1.15$. Hence, $P_{2017}=200\ 000\times1.15=230\ 000$

b $P_{2020}=200000\times1.15^4=349801$

c No. It's too fast. In just 4 years the population almost doubled!

2 $P_{2019} = 2.2 \times (1+0.0265)^4 = \$2.44$

3 $P_6 = P_0(1-p)^6 = 45000 \times (1-0.05)^6 = €33079$

4 **a** $P_3 = 15000 \times (1-0.12)^3 = €10222$

b $€5000 = 15\ 000 \times (0.88)^y$ from the GDC $y = 8.59$. Hence, it would take 8.59 years.

5 **a** $r = 1 + 0.125 = 1.125$

b $r = 1 - 0.073 = 0.927$

c $r = 0.89$

d $r = 1 + 0.001 = 1.001$

6 Here we consider the initial year as the year 0.

a $P_7 = 1.2 \times 1.05^7 = 1.69$ m, % change is applied 7 times

b The 7th measurement is 6 years later, so $P_6 = 1.2 \times 1.05^6 = 1.61$ m, % change is applied 6 times

c We can say that 8 years have passed, so $P_8 = 1.2 \times 1.05^8 = 1.77$ m, % change is applied 8 times

7 **a** $P_{2029} = 7.7 \times 1.0172^{10} = 9.13$ billion people

b $P_{2019} = 7.7 = P_{2012} \times 1.0172^7 \Rightarrow P_{2012} = 7.7 \times 1.0172^{-7} = 6.83$ billion people.,

c We treat the end of 2040 the same as the beginning of 2041. Hence, $P_{2041} = 7.7 \times 1.0172^{22} = 11.21$ billion people.,

d $2 \times 7.7 = 7.7 \times 1.0172^y$ years. From the GDC $y = 40.64 \approx 41$, so population would have doubled by the beginning 2060, i.e. during 2059.

e 2090 is 71 years from 2019, so $10 = 7.7 \times (1+p)^{71} \Rightarrow p = \sqrt[71]{\frac{10}{7.7}} - 1 = 0.003688$. The annual rate should be 0.37% .

Exercise 7C

1 **a** $u_7 = 1 \times 1.4^6 = 7.53$ cm

b The total material is just the sum of the parts, so $S_7 = 1 \times \frac{1.4^7 - 1}{1.4 - 1} = 23.85$ cm will be used.

2 **a** Here, treating "millions" as the units, $u_1 = 12$ and $r = 1.1$, therefore $u_6 = u_1 r^5 = 12 \times 1.1^5 \approx 19.3$. That is, 19 million were created in the sixth hour.

b $S_6 = \frac{u_1(r^n - 1)}{(r-1)} = \frac{12(1.1^6 - 1)}{(1.1 - 1)} = 92.6$ million

c $S_{10} = \dfrac{u_1(r^n - 1)}{(r-1)} = \dfrac{12(1.1^{10} - 1)}{(1.1-1)} \approx 191$ million

$B_{10} = 72 + S_{10} = 72 + 191 = 263$ million

d Since 1 billion = 1000 million, $B_n > 1000$ or $S_n > 928$

$$S_n = \frac{u_1(r^n - 1)}{(r-1)} = \frac{12(1.1^n - 1)}{(1.1-1)} = 120(1.1^n - 1)$$

$120(1.1^n - 1) > 928$

From the GDC $n > 22.7$ so number of complete hours is 23

3 **a** $r = 1 + 0.062 = 1.062$

b Treating "billion" as the units, $E_{2019} = 1.7 \times 1.062^{11} = 3.29$ billion

c $\%_{2008-2019} = \dfrac{3.29 - 1.7}{1.7} \times 100\% = 94\%$

d We look for the number y of years such that

$1.7 \times 1.062^y > 3$.From the GDC $y = 9.44$ years This will happen after 10 years in 2018.

e Soon, almost everyone in the world will have an email address and the number of new email users will not grow as fast.

4 **a** We look for $S_{10} = 33 \times S_5 \Rightarrow 3 \times \dfrac{r^{10} - 1}{r-1} = 33 \times 3 \times \dfrac{r^5 - 1}{r-1} \Rightarrow r^{10} = 33\left(r^5 - 1\right) + 1$

$\Rightarrow r^{10} - 33r^5 + 32 = 0$. This can be solved directly on the GDC or note that we can transform this into a quadratic equation $(r^5)^2 - 33r^5 + 32 = 0$, so

$r^5 = \dfrac{33 \pm \sqrt{33^2 - 4 \times 32}}{2} = 32$ or 1. As the sequence is geometric then $r \neq 1$, so $r = \sqrt[5]{32} = 2$

b We look for $u_n = 3 \times 2^{n-1} < 1000$, From the GDC $n < 9.38$so $u_9 = 3 \times 2^8 = 768$ is the last term below 1000.

c $S_n = 3 \times \dfrac{2^n - 1}{2-1} > 1000$ $n > 8.39$. So $S_9 = 3 \times \dfrac{2^9 - 1}{2-1} = 1533$ is the lowest sum greater than 1000.

d $S_k = 33825 S_5$, where $S_5 = 3 \times \dfrac{2^5 - 1}{2-1} = 93$, so

$S_k = 33\,825 \times 93$ from the GDC $k = 20$

5 **a** We get the ratio $r = \dfrac{10}{2} = 5$. So .$u_k = 2 \times 5^{k-1} = 781\,250$ from the GDC $k = 9$

Hence, the sum up to u_k is $S_9 = 2 \times \dfrac{5^9 - 1}{5-1} = 976\,562$.

b $r = \dfrac{9.6}{6.4} = 1.5 \Rightarrow u_k = 6.4 \times 1.5^{k-1} = 164.025$. From the GDC $k = 9$

Hence, the sum up to u_k is $S_9 = 6.4 \times \frac{1.5^9 - 1}{1.5 - 1} = 479.275$.

c $u_n = \frac{8}{3} \times \left(\frac{1}{4}\right)^{n-1} = \frac{1}{6144}$. From the GDC $n = 8$

Hence, the sum is $S_8 = \frac{8}{3} \times \frac{\frac{1}{4}^8 - 1}{\frac{1}{4} - 1} = \frac{21845}{6144} = 3.56$

6 **a** $r = 1 - 0.002 = 0.998$

b $t_5 = 30.4 \times 0.998^4 = 30.2$ seconds

c After 20 days we expect her to do $t_{20} = 30.4 \times 0.998^{19} = 29.27 < 29.3$ seconds. She will make it that day!

Exercise 7D

1 **a** $r = \frac{0.002}{0.001} = 2 > 1$. Hence, the series diverges.

b $r = \frac{500\ 000}{1\ 000\ 000} = \frac{1}{2} < 1 \Rightarrow S_\infty = \frac{1\ 000\ 000}{1 - \frac{1}{2}} = 2\ 000\ 000$

c $r = \frac{-3}{1} = -3$, $|r| > 1$. Hence, the series diverges.

d $r = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2} < 1 \Rightarrow S_\infty = \frac{\frac{1}{2}}{\frac{1}{2}} = 1$

2 **a** $S_\infty = 7u_1 \Rightarrow \frac{u_1}{1 - r} = 7u_1 \Rightarrow r = 1 - \frac{1}{7} \to r = \frac{6}{7}$

b

$$S_k > \frac{1}{2} S_\infty$$

$$\frac{u_1\left(1 - \left(\frac{6}{7}\right)^k\right)}{1 - \frac{6}{7}} > \frac{\frac{1}{2}u_1}{1 - \frac{6}{7}}$$

$$1 - \left(\frac{6}{7}\right)^k > \frac{1}{2}$$

$$\left(\frac{6}{7}\right)^k < \frac{1}{2}$$

$$k > 4.50$$

So, the sum of the first 5 terms is the lowest sum that is more than half of S_∞.

3 **a** $S_\infty = \dfrac{4}{1-r} = 24 \Rightarrow r = \dfrac{5}{6}$

b $u_1 = 4, u_2 = 4 \times \dfrac{5}{6} = \dfrac{10}{3}, u_3 = 4 \times \left(\dfrac{5}{6}\right)^2 = \dfrac{25}{9}$

c $v_1 = 16, \quad v_2 = \dfrac{100}{9}, \quad v_3 = \dfrac{625}{81}$

d $r_{new} = \dfrac{100}{9 \times 16} = \dfrac{25}{36} = \left(\dfrac{5}{6}\right)^2 = r^2$

e $S_{\infty(new)} = \dfrac{16}{1 - \dfrac{25}{36}} = \dfrac{576}{11}$

f $\dfrac{S_{\infty(new)}}{S_\infty^2} = \dfrac{\dfrac{576}{11}}{24^2} = \dfrac{1}{11}$. The series of squared terms is much less than the squared series of original terms.

4 **a** $B_1 = 2 \times 0.75 = 1.5$ m

b $B_2 = 1.5 \times 0.75 = 1.125$ m

c $B_3 = 1.125 \times 0.75 = 0.84375$ m

d $r = 0.75$

e No, because $r \times (0.75)^k \neq 0$ for any finite $k > 0$.

f $S_\infty = \dfrac{2}{1 - 0.75} = 8$ m

5 **a** $a_1 = \dfrac{\pi 4^2}{2} = 8\pi$ cm^2

b $a_2 = \dfrac{\pi 4^2}{4} = 4\pi$ cm^2

c $S_2 = 8\pi + 4\pi = 12\pi$ cm^2

d The area is halved each time, hence $r = \dfrac{1}{2}$. Thus, $a_5 = 8\pi \times \left(\dfrac{1}{2}\right)^4 = \dfrac{\pi}{2}$ cm^2

e $S_5 = 8\pi \dfrac{\left(\dfrac{1}{2}\right)^5 - 1}{\dfrac{1}{2} - 1} = 15.5\pi$ cm^2

f

$$S_k > 0.99 \times 16\pi$$

$$0.99 \times 16\pi < 8\pi \frac{1-\left(\frac{1}{2}\right)^k}{\frac{1}{2}}.$$

$$\left(\frac{1}{2}\right)^k > 1 - 0.99$$

$$k > 6.64$$

Hence, 7 students are needed to cover at least 99% of the circle.

g $S_\infty = \dfrac{8\pi}{1-\frac{1}{2}} = 16\pi$ cm^2, which is the total area of the circle.

6 **a** It's geometric, so $r = 0.89$, thus $u_3 = 1 \times 0.89^2 = 0.792$ m

b

$$u_k = 1 \times 0.89^{k-1} < 0.5$$
$$k > 6.9$$

After 7 oscillations, he will need to push the swing again.

c $S_7 = \dfrac{0.89^7 - 1}{0.89 - 1} = 5.07$ m

d $S_\infty = \dfrac{1}{1-0.89} = 9.09$ m

Exercise 7E

Note: When using the Finance app if there are no payments into the account beyond the initial payment then set PMT as 0 and P/Y as 1. In this case N will give the answer as the number of years. If the P/Y is set as other than 1 the value of N on most calculators will be the number of payments rather than the number of years.

1 In 15 years:

$$FV_{Oswald} = 5000(1+0.037)^{15} = €8622.86$$

$$FV_{Martha} = 5000\left(1+\frac{0.035}{12}\right)^{12\times15} = €8445.84$$

Oswald will have more.

2 **a** $FV = 50000(1+0.032)^{10} = 68512.05$ NIS

or

```
N=10
I%=3.2
PV=-50000
PMT=0
▪FV=68512.05232
P/Y=1
C/Y=1
```

b

```
▪N=22.00560358
 I%=3.2
 PV=-50000
 PMT=0
 FV=100000
 P/Y=1
 C/Y=1
```

22 years.

c

```
 N=20
▪I%=3.526492384
 PV=-50000
 PMT=0
 FV=100000
 P/Y=1
 C/Y=1
```

This corresponds to a nominal annual interest of 3.53% .

3 a

```
 N=5
▪I%=5.099998897
 PV=-4500
 PMT=0
 FV=5803.94
 P/Y=1
 C/Y=12
```

She has an annual interest rate of 5.1% compounded monthly.

b A 50% increase means for her to have $4500 + 4500 \times 0.5 = £6750$. Thus,

```
▪N=7.967180396
 I%=5.099998897
 PV=-4500
 PMT=0
 FV=6750
 P/Y=1
 C/Y=12
```

She would need 8 years to get a 50% increase in the amount invested.

4 Compounded quarterly means that C/Y is equal to 4,

```
▪N=5.456727663
 I%=7.5
 PV=-1000
 PMT=0
 FV=1500
 P/Y=1
 C/Y=4
```

He will be able to buy it after 6 years.

5 $6762.56 = PV\left(1+\dfrac{0.012}{2}\right)^{2\times10} \Rightarrow PV = \dfrac{6762.56}{1.006^{20}} = €6000$

6 **a** $k = 4$, In one year the rate would be $\left(1+\dfrac{0.06}{4}\right)^{4\times 1} = 1.0614$. Hence, the effective yearly interest rate is of 6.14%.

This can also be calculated using the Finance app. Set PV as -100 and N as 1 year. The final value will give the amount the 100 has increased by from which the effective interest can be calculated.

```
 N=1
 I%=6
 PV=-100
 PMT=0
▪FV=106.1363551
 P/Y=1
 C/Y=4
```

100 has increased to 106.14 hence a 6.14% increase.

b $k = 12$, In one year the rate would be $\left(1+\dfrac{0.06}{12}\right)^{12\times 1} = 1.0617$. Hence, the effective yearly interest rate is of 6.17%.

or

```
 N=1
 I%=6
 PV=-100
 PMT=0
▪FV=106.1677812
 P/Y=1
 C/Y=12
```

7 **a** Adjusting for inflation, $r \Rightarrow r - i = 0.3\%$

b $FV = 20000\left(1+\dfrac{0.003}{12}\right)^{12\times 1} = \$20060$

8 **a** Adjusting for inflation $r \Rightarrow r - i = 0.6\%$

b

$$FV = 2000(1+0.006)^{1} = \$2012$$

or

```
 N=1
 I%=0.6
 PV=-2000
 PMT=0
▪FV=2012
 P/Y=1
 C/Y=1
```

Exercise 7F

1 **a** N is the number of payments and the payments are made monthly. In 10 years there are 120 months so N=120

```
N=120
I%=6
PV=10000
▪PMT=-111.0205019
FV=0
P/Y=12
C/Y=12
```

The payment per month is $111.02

b The balance can be found either by changing N to 60 (for 5 years) or by using the balance function in your finance app. If using the full value obtained for the payment in part a (111.0205…), the solution is $5742.60, but as the payments can only be for whole number of cents $11.02 should be used to give $5742.63.

Both solutions would be acceptable in an examination.

2 a $N = 6 \times 12 = 72$

```
N=72
I%=6.3
PV=40000
▪PMT=-667.9725942
FV=0
P/Y=12
C/Y=4
```

Payments are 667.97AED per month

b At the end of three years $N = 3 \times 12 = 36$ The final value with give the balance which is equal to 21869.79AED

3 a Payments are monthly so N is the number of months: $N = 25 \times 12 = 300$

```
N=300
I%=8
PV=0
PMT=-1000
▪FV=957366.5705
P/Y=12
C/Y=12
PMT:END BEGIN
```

The value of the investment after 25 years is TRY 957366.57

b Interest per month is $951026.39 \times \dfrac{8}{12 \times 100} = 6340.17$, as $6340.17 > 1200$ the money gained by the account each month is much greater than the money being paid out so the money will never run out.

4 a

```
N=72
I%=5.19
PV=11000
▪PMT=-178.1253534
FV=0
P/Y=12
C/Y=12
PMT:END BEGIN
```

Payment is €178.13 per month

b $N = 12 \times 5 = 60$

```
N=60
I%=5.19
PV=11000
PMT=-178.1253534
▪FV=-2078.60708
P/Y=12
C/Y=12
PMT:END BEGIN
```

Money needing to be paid off is €2078.61

If payment is adjusted to €178.13 which would be the actual amount paid the amount owing would be €2078.29

5

```
N=72
I%=14.06
PV=33560
▪PMT=-692.607297
FV=0
P/Y=12
C/Y=12
PMT:END BEGIN
```

The monthly repayment amount is $692.61

As he can pay $\$700$, then he can afford to buy the car.

Exercise 7G

1 **a** $f(x) = 4^x + 1$

i Crosses y -axis at $(0, 1+1) = (0, 2)$

ii Horizontal asymptote at $y = 1$

iii Increasing as $4 > 1$

b $f(x) = 0.2^x - 3$

i Crosses y -axis at $(0, 1-3) = (0, -2)$

ii Horizontal asymptote at $y = -3$

iii Decreasing as $0.2 < 1$

c $f(x) = 5^{2x}$

i Crosses y -axis at $(0,1)$

ii Horizontal asymptote at $y = 0$

iii Increasing as $5 > 1$

d $f(x) = 3^{0.1x} + 2$

i Crosses y -axis at $(0,1+2) = (0,3)$

ii Horizontal asymptote at $y = 2$

iii Increasing as $3 > 1$

e $f(x) = 3 \times 2^{x} - 5$

i Crosses y -axis at $(0,3-5) = (0,-2)$

ii Horizontal asymptote at $y = -5$

iii Increasing as $2 > 1$

f $f(x) = 4 \times 0.3^{2x} + 3$

i Crosses y -axis at $(0,4+3) = (0,7)$

ii Horizontal asymptote at $y = 3$

iii Decreasing as $0.3 < 1$

g $f(x) = 5 \times 2^{0.5x} - 1$

i Crosses y -axis at $(0,5-1) = (0,4)$

ii Horizontal asymptote at $y = -1$

iii Increasing as $2 > 1$

h $f(x) = 2 \times 2.5^{-x} - 1 = 2 \times \left(\frac{1}{2.5}\right)^{x} - 1$

i Crosses y -axis at $(0,2-1) = (0,1)$

ii Horizontal asymptote at $y = -1$

iii Decreasing as $\frac{1}{2.5} < 1$

2 a $2 \times 4^{2x} + 5 = 2\left(4^{2}\right)^{x} + 5 = 2 \times 16^{x} + 5$

b $7 \times 0.5^{-3x} + 2 = 7\left(\left(\frac{1}{2}\right)^{-3}\right)^{x} + 2 = 7 \times 8^{x} + 2$

3 **a** $S(0) = 12 + 10 \times 1 = 22$

b We want that $S(t) = 15 = 12 + 10 \times 1.2^{-t} \to t = 6.6$ hours from the GDC.

c It's a translation of $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$. Thus, $S_2(t) = 14 + 10 \times 1.2^{-t}$

d It's a stretch parallel to the S axis of 2. Thus $S_3(t) = 12 + 10 \times 1.2^{-\frac{t}{2}}$

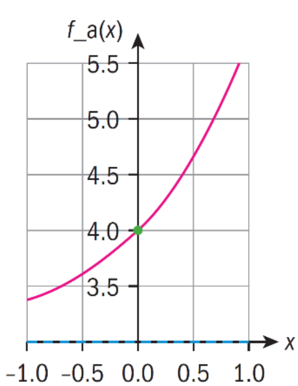
e It's a stretch parallel to the t axis of 2. Thus $S_4(t) = 24 + 20 \times 1.2^{-t}$

Exercise 7H

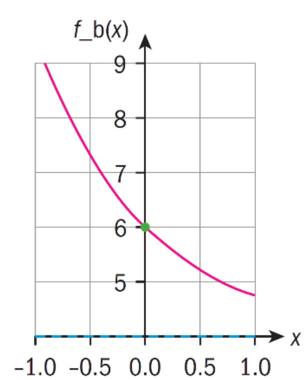
1

	y-intercept	Horizontal asymptote	Growth or decay	Range
$f(x) = e^{x} + 3$	$(0, 4)$	$y = 3$	Growth	$]3, \infty[$
$f(x) = 2e^{-x} + 4$	$(0, 6)$	$y = 4$	Decay	$]4, \infty[$
$f(x) = 0.2e^{0.3x} - 2$	$(0, -1.8)$	$y = -2$	Growth	$]-2, \infty[$
$f(x) = 5 - 2e^{-3x}$	$(0, 3)$	$y = 5$	Decay	$]-\infty, 5[$

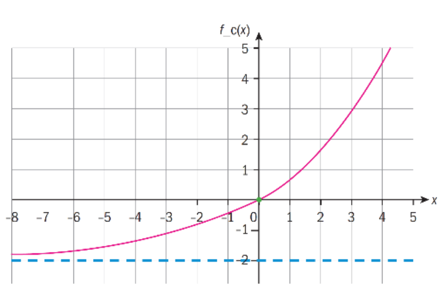
2 **a**



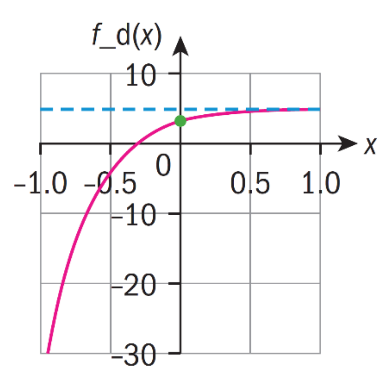
b



c



d



3 $T(t) = 24.5 + 55.9e^{-0.0269t}$

a Crosses T-axis at $(0, 80.4)$

b The y-intercept is the temperature of the water when it is initially poured into the cup.

c It's decay as the coefficient of the exponent is negative.

d The temperature decays over time as the water cools.

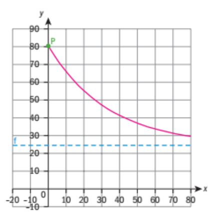
e $y = 24.5$

f The horizontal asymptote represents room temperature. After a long time has passed, the water will be in equilibrium with the room.

g $24.5 < T \le 80.4$

The upper bound is the temperature of the water when it was initially poured into the cup and the lower bound is the equilibrium temperature after a long time.

h



4 **a** The piecewise function must have the same value for $x = 3$ on both sides to be continuous. Thus, $a \times (3)^3 = 2e^{(3)} \Rightarrow a = 2\left(\frac{e}{3}\right)^3 = 1.488$.

b The piecewise function must have the same value for $x = 4$ on both sides to be continuous. Thus, $a \times (4)^2 + 8 = 2^{(4)+1} \Rightarrow a = \frac{24}{16} = \frac{3}{2}$

Exercise 7I

1 **a** $n(23) = 252 \Rightarrow 252 = 8e^{k\times 23} \Rightarrow k = 0.15$

b One hour is 60 minutes, so $n(60) = 8e^{0.15\times 60} = 64825$ bacteria.

c $n(T) > 100\ 000 \Rightarrow 8 \times e^{0.15\times T} > 100\ 000$ minutes.

Using the table (or graph / solver) function in the calculator we find that in 63 minutes there will be over 100 000 bacteria.

2 **a** Initial mass when $t = 0$. Thus, $m(0) = 952$ kg.

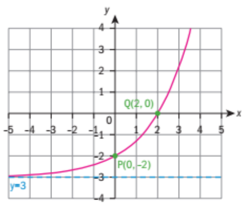
b $m(2.5) = 952e^{-\frac{2.5}{3}} = 414$ kg.

c $m(t_{half}) = \frac{1}{2} \times 952 \Rightarrow e^{-\frac{t_{half}}{3}} = \frac{1}{2} \Rightarrow t_{half} = 2.08$ hours from the GDC.

d $m(t_{10\%}) = 0.1 \times 952 \Rightarrow t_{10\%} = 6.91$ hours.

3 a $y = -3$

b



c The domain of the inverse function is the range of the original function and vice versa. Hence domain is $x > -3$ and the range is $f^{-1}(x) \in$.

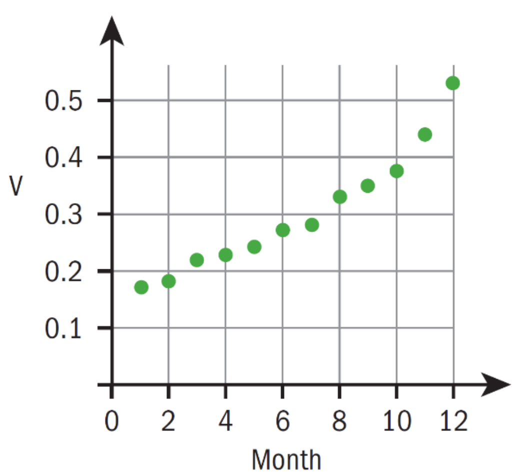
4 a As initially there were 5000 ml, then $a = 5000$.

b $V\left(\frac{35}{60}\right) = 151 = 5000e^{b\times 35} \Rightarrow b = -0.100$

c $V(T) < 100 \Rightarrow 5000e^{-0.10T} < 100$. From the table function (or graph / solver) on the calculator $T \geq 40$ so the least value of T is 40 minutes..

5 a

Month	1	2	3	4	5	6
Value (CZK) millions	7.17	7.18	7.21	7.22	7.24	7.27
Month	7	8	9	10	11	12
Value (CZK) millions	7.27	7.33	7.35	7.38	7.44	7.52



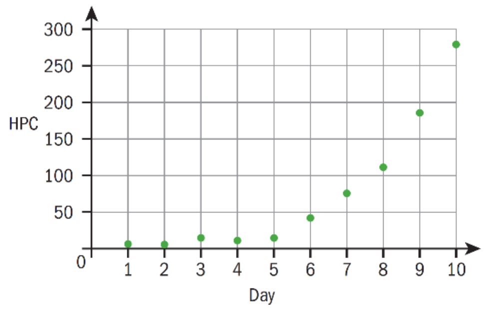
b It's an increasing set of points with increasing differences, possibly approaching an asymptote at $v=0$

c Using calculator, $v = 0.144(1.11)^t$

d Using the calculator $R^2 = 0.998$

e. $v = 0.150(1.10)^t > 2 \Rightarrow t > 25.2$, so 26 months or about 2 years and 2 months.

6 a

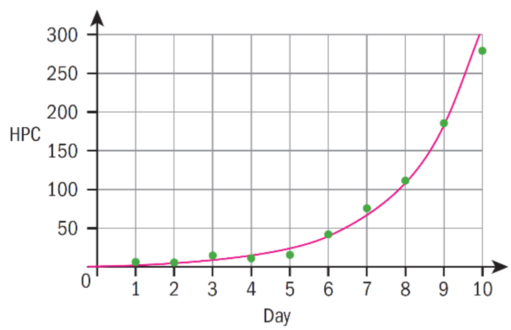


b HPC and the rate of change are constantly increasing.

c From the GDC $HPC(t) = 2.67(1.59)^t$

d Using the calculator, $R^2 = 0.998$

e



f From the table, HPC > 50 for the first time on day 7. Therefore, he should apply the disinfectant every 7 days.

7 a $N = 1000 \times 2^0 = 1000$

b $4000 = 1000 \times 2^{4k}$

$2^{4k} = 4 = 2^2$

$4k = 2$

$k = \dfrac{1}{2}$

c $N = 1000 \times 2^4 = 16000$

d $32000 = 1000 \times 2^{\frac{1}{2}t}$

$2^{\frac{1}{2}t} = 32 = 2^5$

$\dfrac{1}{2}t = 5$

$t = 10$ days

Exercise 7J

1 **a** $\log(100) = \log(10^2) = 2$

b $\log(0.1) = \log\left(\frac{1}{10}\right) = \log(10^{-1}) = -1$

c $\ln(e) = 1$

d $\ln\left(\frac{1}{e}\right) = \ln(e^{-1}) = -1$

e $\ln(e^2) = 2$

f $\log(\sqrt{10}) = \log\left(10^{\frac{1}{2}}\right) = \frac{1}{2}$

g $\ln\left(\frac{1}{\sqrt{e}}\right) = \ln\left(e^{-\frac{1}{2}}\right) = -\frac{1}{2}$

2 **a** $x = \log(10) = 1$

b $x = \log(100) = 2$

c $x = \log(38)$

d $x = \ln(e^2) = 2$

e $x = \ln(3)$

f $x = \ln(0.3)$

g $x = \ln(1) = 0$

3 **a** $\log(x^2)$

b $\log\left(x^{\frac{1}{3}}\right) = \log(\sqrt[3]{x})$

c $\log(x^3y^2)$

d $\log\left(\frac{x}{y^3}\right)$

e $\ln(x^{-2}) = \ln\left(\frac{1}{x^2}\right)$

f $\log(10^2) + \log(x) = \log(100x)$

g $\ln\left(\frac{x}{yz}\right)$

4 a The piecewise function must have the same value for $x = 1$ on both sides to be continuous. Thus, $4\times(1)^3 - 3 = 2e^{a\times(1)} \rightarrow a = \ln\left(\frac{1}{2}\right)$.

b The piecewise function must have the same value for $x = 2$ on both sides to be continuous. Thus, $3\times(2)^2 - 4 = 2\ln\left(a\times(2)\right) \rightarrow a = \frac{1}{2}e^4$.

Exercise 7K

1 a i $x = \log(3)$

ii $x = \log(75)$

iii $x = \ln(5)$

b i $e^x = \frac{15}{3} \Rightarrow x = \ln\left(5\right)$

ii $10^x = 2\times 2 \Rightarrow x = \log\left(4\right)$

iii $e^x = \frac{5}{3} + 1 \Rightarrow x = \ln\left(\frac{8}{3}\right)$

2 a $x = \log_a(c)$

b $x = \ln\left(\frac{2}{b}\right)$

c $x = \log\left(\frac{k}{2}\right)$

3 a $10^{3\log(x)} = 10^{\log(x^3)} = x^3$

b $e^{\ln(x)-\ln(y)} = e^{\ln\left(\frac{x}{y}\right)} = \frac{x}{y}$

c $10^{2\log(x)-\log(y)} = 10^{\log\left(\frac{x^2}{y}\right)} = \frac{x^2}{y}$

d $e^{-2\ln(x)} = e^{\ln\left(\frac{1}{x^2}\right)} = \frac{1}{x^2}$

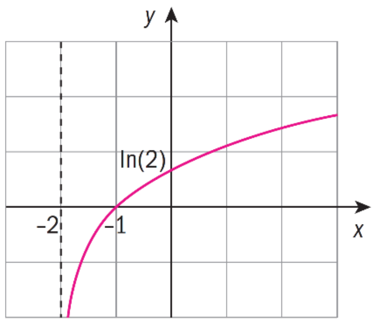
4 a $g\left(f(x)\right) = 2e^{\ln(x+2)} = 2(x+2) = 2x + 4$

b Vertical asymptote at $x = -2$

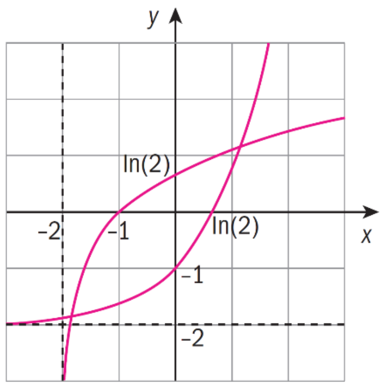
Intersection with y -axis at $\left(0, \ln(2)\right)$

Intersection with x -axis at $\ln\left(x+2\right) = 0 \Rightarrow x = -1$. Hence, it is at $(-1, 0)$

c



d



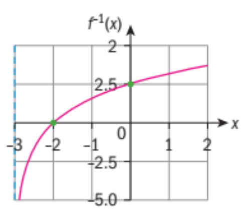
Hence, $f^{-1}(x)=e^{x}-2$

5 **a** $y=10^{x}-3 \Rightarrow x=\log(y+3)$, therefore $f^{-1}(x)=\log(x+3)$

b Vertical asymptote: $x=-3$

Intersection y -axis: $(0,\log(3))$

Intersection x -axis: $\log(x+3)=0 \Rightarrow x=-2$, therefore $(-2,0)$



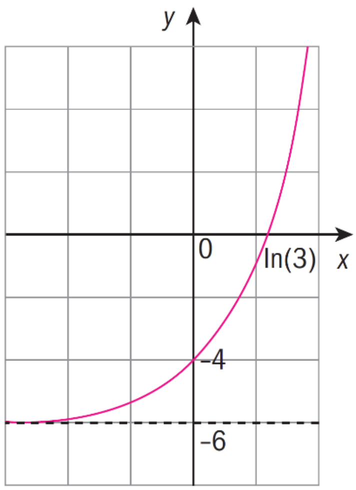
c The domain is from the asymptote onwards, therefore $x>-3$

The range is all real, so $f^{-1}(x)\in\mathbb{R}$.

6 **a** Horizontal asymptote: $y=-6$

Intersection y -axis: $(0,4)$

Intersection x -axis: $2e^{x}-6=0 \Rightarrow x=\ln(3)$, therefore $(\ln(3),0)$



b $y = 2e^x - 6 \Rightarrow x = \ln\left(\frac{1}{2}(y+6)\right)$, therefore $f^{-1}(x) = \ln\left(\frac{1}{2}(x+6)\right)$

c The vertical asymptote is at $x = -6$, so the domain is $x > -6$. The range is all the real numbers, so $f^{-1}(x) \in \mathbb{R}$.

Exercise 7L

1 a $B = \log y = 3.2 \Rightarrow y = 10^{3.2} = 1585$

$G = \log y = 6.6 \Rightarrow y = 10^{6.6} = 3981072$

G is 2512 times greater than B.

b It goes from $A \to y = 10^{1.7} = 50.12$ to $J \to y = 10^{10.1} = 1.259 \times 10^{10}$.

c Using a log scale reduces the range and spreads the data points move evenly.

2 a

Country	GNI	LE	log(GNI)	log(LE)
AUT	42080	80.7	4.62408	1.90687
BEL	39270	80.2	4.59406	1.90417
BGR	13980	74.2	4.14551	1.8704
HRV	19330	77.1	4.28623	1.88705
CZE	24280	77.8	4.38525	1.89098
CNK	42300	79.9	4.62634	1.90255
EST	20830	76.3	4.31869	1.88252
FIN	38500	80.3	4.58546	1.90472
FRA	35650	81.6	4.55206	1.91169

DEU	39970	80.5	4.60173	1.9058
GRC	26090	80.5	4.41647	1.9058
HUN	20260	75	4.30664	1.87506
IRL	33230	80.6	4.52153	1.90634
ITA	32710	82	4.51468	1.91381
LVA	17820	74.6	4.25091	1.87274
LTU	19690	74.3	4.29425	1.87099
LUX	64410	81.5	4.80895	1.91116
NLD	43260	80.9	4.63609	1.90795
POL	20480	76.5	4.31133	1.88366
PRT	24480	80.2	4.38881	1.90417
ROU	15140	74.5	4.18013	1.87216
SVK	22230	76.1	4.34694	1.88138
SVN	26960	79.9	4.43072	1.90255
ESP	31660	82	4.50051	1.91381
SWE	42200	81.7	4.62531	1.91222
GBR	35940	80.5	4.55558	1.9058

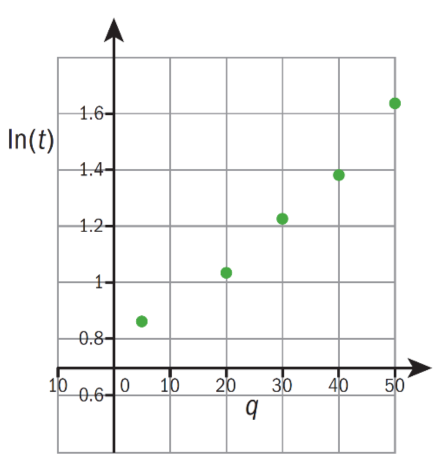
b $y = ax^b \Rightarrow \log y = b\log x + \log a = 0.0795\log x + 1.54$

c In this linear model we get $\log a = 1.54 \Rightarrow a = 34.67$, $b = 0.0795$

d In the power model we get the same values, with any differences due to rounding during the process.

3 **a** $\ln t = \{0.875, 1.03, 1.22, 1.39, 1.63\}$

b



c $t = ae^{bq} \Rightarrow \ln t = bq + \ln a$

From the linear regression function on the GDC ln$a$=0.742 and b=0.0168

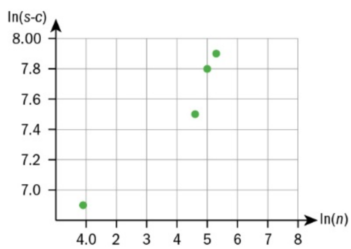
d $\ln a = 0.744 \Rightarrow a = e^{0.744} = 2.10$ giving the final model: $t = 2.10e^{0.0168q}$

e $t(60) = 2.10e^{0.0168\times60} = 5.75$ minutes

4 a $s(0) = c = \$1000$

b $s - c = a \times n^b \Rightarrow \ln(s-c) = b\ln n + \ln a$

c



d $\ln(a) = 3.99, b = 0.75$

e $\ln(s-1000) = 3.99 + 0.75\ln n \Rightarrow s(n) = 54.0 \times n^{0.75} + 1000$

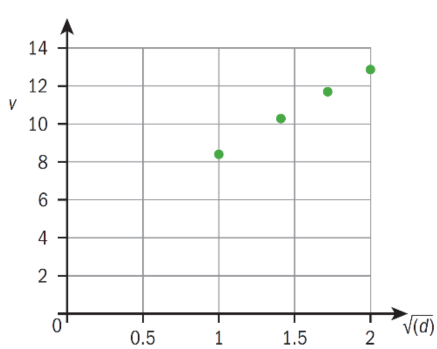
f $S(500) = \$6710$

g The data only go up to 200 shirts; therefore, this is extrapolation far beyond the original data.

h i $x = n,\ y = S - c$

ii This gives the same values for a and b. Any differences are due to rounding during the calculations.

5 a



b We can get a system of equations such that:

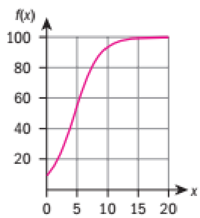
$8.4 = a\sqrt{1} + b \Rightarrow a = 8.4 - b$

$12.9 = a\sqrt{4} + b \Rightarrow b = 12.9 - 2a$

Substituting for b : $a = 8.4 - 12.9 + 2a \Rightarrow a = 4.5$ and $b = 12.9 - 2 \times 4.5 = 3.9$

Exercise 7M

1 a



b When $t = 0 \Rightarrow P(0) = \dfrac{100}{11} = 9.09\%$, which is the initial recorded percent of people with access to the Internet.

As $t \to \infty, P(t) \to \dfrac{100}{1} = 100\%$.

Hence range is $9.09 \le P(t) < 100$

This means that we are tending asymptotically to a state where everyone will have the Internet, at least according to the model.

c We look for $P(t_{half}) = 50 = \dfrac{100}{1+10e^{-0.5t_{half}}} \Rightarrow t_{half} = -\dfrac{1}{0.5}\ln\left(\dfrac{\frac{100}{50}-1}{10}\right) = 4.6$ years.

d $P(20) = \dfrac{100}{1+10e^{0.5\times 20}} = 99.95\%$

2 a In 1950 there are $P(0) = \dfrac{107000}{5} = 21400$ people.

b In 2015 there are $P(65) = \dfrac{107000}{1+4e^{-0.135\times 60}} = 106934$ people.

c The limit population is when $t \to \infty,\ P(t) \to \dfrac{107\ 000}{1+0} = 107\ 000$ people.

3 a The behaviour at infinite should be 120 million so as $t \to \infty,\ P(t) \to L = 120\ 000\ 000 = 120 \times 10^6$.

Initially there are 10000 people infected, so $P(0) = \dfrac{120\times 10^6}{1+C} = 10\ 000 \Rightarrow C = 11\ 999$.

After two weeks there are 20000 people infected, so $P(2) = \dfrac{120\times 10^6}{1+11999\times e^{-k\times 2}} \to$

$k = -\dfrac{1}{2}\ln\left(\dfrac{1}{11999}\left(\dfrac{120\times 10^6}{20000} - 1\right)\right) = 0.347$.

Therefore, $f\left(x\right)=\dfrac{120\times10^6}{1+11999\times e^{-0.347x}}$.

b $f\left(4\right)=40000$ infected to 2 significant figures.

c $P\left(x_{90\%}\right)=0.9\times120\times10^6=\dfrac{120\times10^6}{1+11\ 999\times e^{-0.347x_{90\%}}}\Rightarrow x_{90\%}=-\dfrac{1}{0.347}\ln\left(\dfrac{1}{11\ 999}\left(\dfrac{1}{0.9}-1\right)\right)=33.4$ weeks.

Chapter Review

1 **a** $r=\dfrac{5}{2}=2.5$

b $u_8=2\times2.5^7=1220.70$

c $S_8=2\times\dfrac{2.5^8-1}{2.5-1}=2033.17$

2 $FV=350\left(1-0.12\right)^5=\$184.71$

3 **a** This is a geometric series since the percentage increase is a constant. Thus,

$\dfrac{439\ 230}{363\ 000}=\dfrac{u_5}{u_3}=\dfrac{u_3\times r^2}{u_3}=r^2\Rightarrow r=\sqrt{\dfrac{439\ 230}{363\ 000}}=1.1$, so the percentage increase is $p=10\%$.

b As it is the beginning of the third year two complete years have passed

$363\ 000=u_0\times1.1^2\Rightarrow u_0=€300000$

c At the beginning of the ninth year, eight complete years have passed

$u_9=300000\times1.1^8=€643\ 077$

4 **a** First, we get the ratio $r=\dfrac{6}{2}=3$. Now, we see which term is 118098 on the sequence:

$u_N=118098=2\times3^{N-1}\to N=\log_3\left(\dfrac{118098}{2}\right)+1=11$. There are 11 terms in the sequence.

b $S_{11}=2\times\dfrac{3^{11}-1}{3-1}=177146$

5 **a** At the beginning of the 2nd year it would cost $9500\left(1+0.0116\right)=£9610.20$

b Starting with $£9610.20$ at the 2nd year, the 3rd year would be

$£9610.20\left(1+0.0114\right)=£9719.76$

6 **a** $P_6=3000\left(1+\dfrac{0.0235}{12}\right)^{12\times6}=\text{SGD}3453.80$

or

```
N=6
I%=2.35
PV=-3000
PMT=0
▪FV=3453.797693
P/Y=1
C/Y=12
```

b $P_N = 5000 = 3000\left(1+\dfrac{0.0235}{12}\right)^{12\times N}$ years. Solving on a GDC gives $N = 21.8$ years

or

```
▪N=21.75853805
I%=2.35
PV=-3000
PMT=0
FV=5000
P/Y=1
C/Y=12
```

Hence $N = 21.8$ years

7 $P_6 = 5179.27 = 4500\left(1+\dfrac{r}{4}\right)^{4\times 6} \Rightarrow r = 4\left(\left(\dfrac{5179.27}{4500}\right)^{\frac{1}{4\times 6}} - 1\right) = 0.0235 = 2.35\%$

or

```
N=6
▪I%=2.349988186
PV=-4500
PMT=0
FV=5179.27
P/Y=1
C/Y=4
```

Interest is 2.35%

8

```
N=60
I%=4
PV=-4000
▪PMT=73.66608822
FV=0
P/Y=12
C/Y=12
PMT:END BEGIN
```

Payments are \$73.67 per month

9

```
N=60
I%=2.5
PV=6500
▪PMT=-115.3578504
FV=0
P/Y=12
C/Y=12
PMT:END BEGIN
```

The payments are 115.36 euros per month.

10 a $f(5) = 24000 \times 1.12^5 = 42296$ rabbits

b $f(t_D) = 48\ 000 = 24\ 000 \times 1.12^{t_D} \Rightarrow t_D = 6.12$ years.

11 a $T(0) = 21 + 74 = 95°C$

b $T(10) = 21 + 74 \times 1.2^{-10} = 32.95°C$

c $T(x_{40}) = 40 = 21 + 74 \times 1.2^{-x_{40}} \Rightarrow x_{40} = 7.46$ minutes.

d The cup of soup will get to room temperature when $x \to \infty \Rightarrow T(x) \to 21 + 0 = 21°C$

12 a $y = 16$

b Cuts at $x = 0$, so $f(0) = 4 + 16 = 20$. Hence, the intersection point is $(0, 20)$.

13 a $y(7) = 4 + e^7 = 1101$ infected people.

b $y(x_{25k}) = 25\ 000 = 4 + e^{x_{25k}} \Rightarrow x_{25k} = \ln(25\ 000 - 4) = 10.13$ days. That is, at least 25000 people are infected after 11 days.

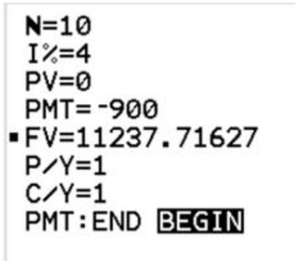
14 a $h(4) = 0.25 + \log(2 \times 4 - 0.6) = 1.12$ m

b $h(t_2) = 2 = 0.25 + \log(2t_2 - 0.6) \Rightarrow t_2 = \frac{1}{2}\left(10^{2-0.25} + 0.6\right) = 28.42$ weeks. That is, after 29 weeks the plant is more than 2 m tall.

15 a $\sum_{r=1}^{n} 5 \times 2^r = 10 \times \frac{2^n - 1}{2 - 1} = 10 \times 2^n - 10 = 10(2^n - 1)$

b $\sum_{r=1}^{2n} 5 \times 2^r = 10 \times \frac{2^{2n} - 1}{2 - 1} = 10(4^n - 1)$

16 Note than on this occasion the payments are made at the start of the time periods not the end.



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Exam style questions

17 a 0.1 kg

b $0.05 = 0.1e^{-0.4t} \Rightarrow t = 1.73$ years

c $0.01 = 0.1e^{-0.4t} \Rightarrow t = 5.76$ years

18 a $\log xy = \log x + \log y = p + q$

b $\log \frac{x}{y} = \log x - \log y = p - q$

c $\log \sqrt{x} = \frac{1}{2} \log x = \frac{1}{2} p$

d $\log x^2 y^5 = 2\log x + 5\log y = 2p + 5q$

e $\log x^y = y \log x = 10^q p$

f $\log 0.01x^3 = \log \frac{x^3}{100} = \log x^3 - \log 10^2 = 3p - 2$

19 a $\frac{a}{b+1} = 2, \ \frac{a}{b} = 3 \Rightarrow a = 6, \ b = 2$

b $h(10) = 2.53$ m (3 sf)

c $t = -10 \ln\left(\frac{a}{h} - b\right)$. For $h(t) = 2.8$ m $\Rightarrow t = 19.5$ years

20 a **i** $t = \frac{-6000}{\ln 2} \ln(0.5) = 6000$ years **ii** $t = \frac{-6000}{\ln 2} \ln(0.25) = 12000$ years

b $t = \frac{-6000}{\ln 2} \ln(0.01) = 40000$ years

c $r = e^{-\left(\frac{\ln 2}{6000}\right)t}$

d $r = e^{-\left(\frac{\ln 2}{6000}\right)10000} = 0.315$ (3 sf)

21 a $ar^2 = 18, \ ar^5 = 486 \Rightarrow r^3 = 27 \Rightarrow r = 3, \ a = 2$

b **i** $u_8 = 2 \times 3^7 = 4374$ **ii** $S_8 = 2\dfrac{(3^8 - 1)}{(3 - 1)} = 6560$

c $2 \times 3^{n-1} > 10^6$,can be solved directly on the GDC or using logs, giving $n > \log_3\left(\dfrac{10^6}{2}\right) + 1 = 12.9$ > Hence, $n = 13$.

22 a **i** $u_n = ar^{n-1}$ **ii** $S_n = a\left(\dfrac{r^n - 1}{r - 1}\right)$

b $v_n = \log\left(ar^{n-1}\right) = \log a + (n-1)\log r$

v_n is an arithmetic progression, with first term $v_1 = \log a$ and common difference $d = \log r$

c

$$T_n = \frac{n}{2}\left(2v_1 + (n-1)d\right)$$
$$= \frac{n}{2}\left(2\log a + (n-1)\log r\right)$$

d $\log S_n = \log\left(a\left(\dfrac{r^n - 1}{r - 1}\right)\right) = \log a + \log\left(r^n - 1\right) - \log\left(r - 1\right)$

$\neq \dfrac{n}{2}\left(2\log a + (n-1)\log r\right)$ so answer is no.

23 a $\dfrac{a}{1-r} = 3a \Rightarrow 1 - r = \dfrac{1}{3} \Rightarrow r = \dfrac{2}{3}$

b $\dfrac{a}{1-r} = \dfrac{2}{3}a \Rightarrow 1 - r = \dfrac{3}{2} \Rightarrow r = -\dfrac{1}{2}$

c $\dfrac{a}{1-r} = \dfrac{a}{3} \Rightarrow 1 - r = 3 \Rightarrow r = -2$ but sum to infinity does not exist in this case as the condition for sum to infinity to exist is $|r| < 1$ So the answer is no.

24 a Require $1.03x = 100 \Rightarrow x = £97.09$

b Require $(1.03)^{10}x = 100 \Rightarrow x = £74.41$

c

```
N=10
I%=3
PV=0
▪PMT=-84.68981224
FV=1000
P/Y=1
C/Y=1
PMT:END BEGIN
```

Her payments will be $M = £84.69$

25 a $5000(1.04)^5 = 6083.26$ euros

$$5000\left(1+\frac{0.038}{12}\right)^{60} = 6044.43 \text{ euros}$$

She should choose Scheme 1.

Part a can also be done using the Finance app on the GDC.

b $6083.26 - 5000 = 1083.26$ euros

c For scheme 1,

$$r_1 = (1.04)^n$$
$$\ln r_1 = n \ln 1.04 = 0.039221n$$

For scheme 2,

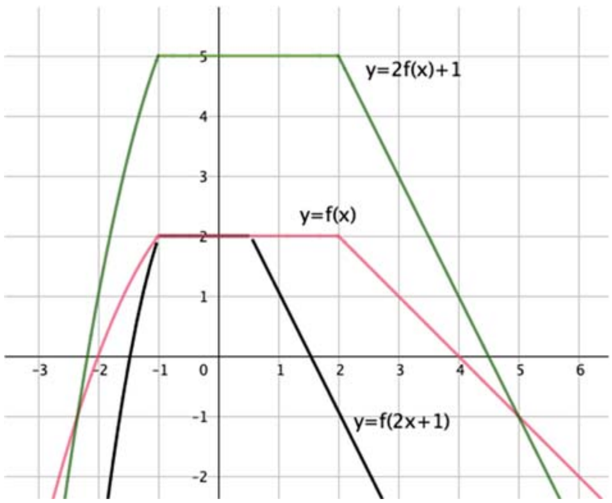
$$r_2 = \left(1+\frac{0.038}{12}\right)^{12n}$$
$$\ln r_2 = 12n \ln 1.0032 = 0.037940n$$

Since $\ln r_1 > \ln r_2$, then scheme 1 is always the best.

8 Modelling periodic phenomena: trigonometric functions and complex numbers

Skills Check

1 a,b



Exercise 8A

1 a $\frac{30°\pi}{180°} = \frac{\pi}{6}$ **b** $\frac{165°\pi}{180°} = \frac{11\pi}{12}$ **c** $\frac{270°\pi}{180°} = \frac{3\pi}{2}$ **d** $\frac{300°\pi}{180°} = \frac{5\pi}{3}$

e $\frac{210°\pi}{180°} = \frac{7\pi}{6}$

2 a $\frac{\pi}{3} \times \frac{180°}{\pi} = 60°$ **b** $\frac{4\pi}{3} \times \frac{180°}{\pi} = 240°$ **c** $\frac{3\pi}{5} \times \frac{180°}{\pi} = 108°$ **d** $3\pi \times \frac{180°}{\pi} = 540°$

e $1 \times \frac{180°}{\pi} = 57.3°$

3 $r = 7\text{m}$ and $\theta = \frac{2\pi}{3}$:

a Using the arc length formula $L = r\theta$, we get that $L = 7 \times \frac{2\pi}{3} = 14.7\text{ m}$

b Using the arc area formula $A = \frac{1}{2}r^2\theta$, we get that $A = \frac{1}{2}7^2 \times \frac{2\pi}{3} = 51.3\text{ m}^2$

c Area $= \frac{1}{2}r^2 \sin\theta = \frac{1}{2}7^2 \sin\frac{2\pi}{3} = 21.2\text{ m}^2$.

d Area of segment = area of sector − area of triangle $= 51.3 - 21.2 = 30.1\text{ m}^2$.

4 a We sum the arc length of the curved part and the two radii $r = 3$ cm to get the total perimeter: $P = 3 + 3 + 3 \times 0.7 = 8.1$ cm.

$A = \frac{1}{2}r^2\theta = \frac{1}{2}3^2 \times 0.7 = 3.15\text{ cm}^2$.

b For the perimeter, the arc AB is given by **3 a**, and the line segment AB is the base of the isosceles triangle AOB, which is $2\times 7\sin\left(\frac{\pi}{3}\right)=12.1\text{ m}$. Hence, the total perimeter is $P=14.7+12.1=26.8\text{ m}$. The area is the same as in **3 d**, so it is 30.1 m^2.

c Arc length of AB: $r\theta=6\times 2=12\text{ cm}$

Perimeter = $12+6+6=24\text{ cm}$.

Area of sector = $\frac{1}{2}r^2\theta=36\text{ cm}^2$

Area of triangle = $\frac{1}{2}6^2\sin(2)=16.4\text{ cm}^2$

Area of segment = $36-16.4=19.6\text{ cm}^2$.

5 **a** As the opposite sides of the rectangle are parallel:

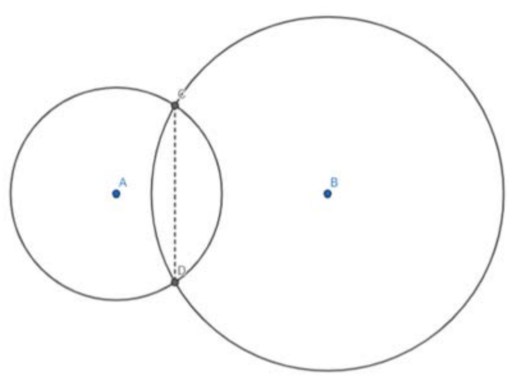
$\alpha=\sin^{-1}\left(\frac{2.5}{3}\right)=56.4°$ or 0.985 radians

b The area of grass is the area of the sector of the circle plus the area of the triangle:

Length of the base of the triangle is $\sqrt{3^2-2.5^2}=1.66$

Area = $\frac{1}{2}\times 1.66\times 2.5+\frac{1}{2}3^2\times 0.985=6.51\text{ m}^2$.

6



Let $C\hat{A}D=\alpha$ and $C\hat{B}D=\beta$.

a AB = 6 cm, AC = 3 cm, BC = 5 cm

For ΔABC,

$$5^2=3^2+6^2-2\times 3\cos\left(\frac{\alpha}{2}\right)$$

$$\alpha=2\cos^{-1}\left(\frac{-20}{-36}\right)=1.96\text{ rad}$$

Similarly, $\beta=1.04\text{ rad}$

$P=r_B\beta+r_A\alpha=11.1\text{ cm}.$

b The area of the region is the area spanned by the arcs ACD and BCD, where $A_{\text{segment}_{ACD}} = A_{\text{sector}_{ACD}} - A_{\text{triangle}_{ACD}}$ and similarly for sector BCD.

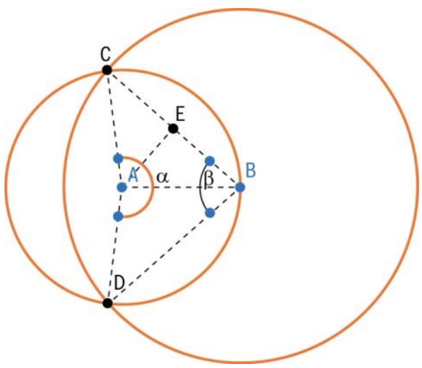
Now, $A_{\text{sector}_{ACD}} = \frac{1}{2} r_A^2 \alpha = 8.82 \text{ cm}^2$, $A_{\text{triangle}_{ACD}} = \frac{1}{2} 3^2 \sin 1.96 = 4.16 \text{ cm}^2$.

Thus, $A_{\text{segment}_{ACD}} = 8.82 - 4.16 = 4.66 \text{ cm}^2$.

Similarly, $A_{\text{segment}_{BCD}} = 13.0 - 10.78 = 2.22 \text{ cm}^2$.

Hence, the total area is $A_{\text{shaded}} = 4.66 + 2.22 = 6.88 \text{ cm}^2$.

7



Note that $\alpha > 180°$. AB = AC = AD = 2 cm. BC = BD = 3 cm.

The shaded area is the sum of the sector CBD of radius 3 cm and angle β plus the two segments for chords CB and BD.

First, we get the angles $\frac{\alpha}{2} = C\hat{A}B = 2 \times \sin^{-1}\left(\frac{BE}{AB}\right) = 2 \times \sin^{-1}\left(\frac{3/2}{2}\right) = 1.70 \text{ rad}$ and $\frac{\beta}{2} = C\hat{B}A = \cos^{-1}\left(\frac{BE}{AB}\right) = \cos^{-1}\left(\frac{3/2}{2}\right) = 0.72 \text{ rad}$.

Thus, the area of sector CBD is simply $\frac{1}{2} \times 3^2 \times (2 \times 0.72) = 6.48$.

Sector CAB is congruent to sector DAB.

$A_{\text{sector}_{CAB}} = \frac{1}{2} \times 2^2 \times 1.70 = 3.40$ and $A_{\text{triangle}_{CAB}} = \frac{1}{2} 2^2 \sin(1.70) = 1.98$. Thus, $A_{\text{segment}_{CAB}} = 3.40 - 1.98 = 1.42$.

Hence the total shaded area is $A_{\text{shaded}} = 6.48 + 2 \times 1.42 = 9.32$.

Exercise 8B

1 **a** 0 **b** 1 **c** 0 **d** 0
e −1 **f** −1 **g** −1 **h** 0

2 **a** There are two solutions per cycle. These sum half a cycle in the case of the sine and a whole cycle in the case of the cosine. As $0 \le x \le 720$, there will be four solutions on each exercise.

i $x = \{10°, 170°, 370°, 530°\}$

ii The first solution is $x = \sin^{-1}(0.3) = 17.5°$. Hence, $x = \{17.5°, 162.5°, 377.5°, 522.5°\}$

iii $x = \{160°, 200°, 520°, 560°\}$

b $0 \le x \le 3\pi$, so there may be 3 or 4 solutions on each exercise.

i $x = \left\{\frac{2\pi}{5}, \frac{3\pi}{5}, \frac{12\pi}{5}, \frac{13\pi}{5}\right\}$

ii $x = \{1.67, 4.61, 7.95\}$

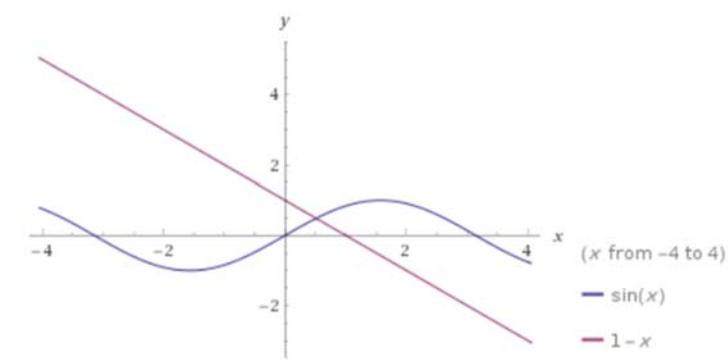
iii $x = \left\{\frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}\right\}$

3 **a** By inspecting the graph, one can find the points of intersection:

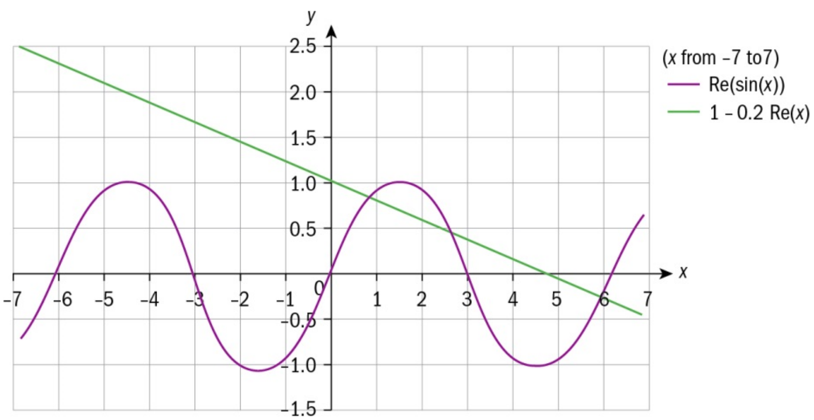
i $(x, y) = (0.51, 0.49)$

ii $(x, y) = \{(0.95, 0.81), (2.65, 0.47), (6.07, -0.21)\}$

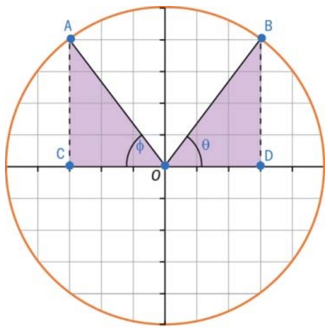
b **i** The graph shows that there is only one solution, because range of $y = \sin x$ is $-1 \le y \le 1$ so there can be no more intersections.



ii The graph shows that there are three solutions, because range of $y = \sin x$ is $-1 \le y \le 1$



4 **a**



The two points with angles θ and $180-\theta$ are shown as the points B and A on the diagram. Point A has coordinates (-x,y) and point B coordinates (x,y). As they have the same *y* coordinate we can determine that , $y=\sin\theta=\sin(180-\theta)$.

b **i** $\sin(\theta)=\frac{1}{2}\Rightarrow\theta=\sin^{-1}\left(\frac{1}{2}\right)=\frac{\pi}{6}\times\frac{180}{\pi}=30°$

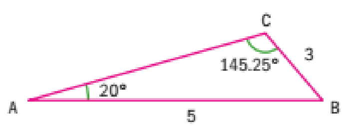
ii $\sin(30°)=\sin(180°-30°)=\sin(150°)=\frac{1}{2}$

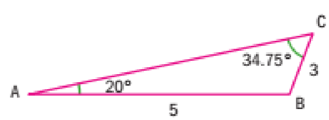
c **i** Using the sine rule: $\frac{\sin(20°)}{3}=\frac{\sin(B\hat{C}A)}{5}\Rightarrow\sin(B\hat{C}A)=\frac{5\sin(20°)}{3}$.

ii $\sin(B\hat{C}A)=0.57\Rightarrow B\hat{C}A=34.8°$.

iii $\sin(34.8°)=\sin(180°-34.8°)=\sin(145.2°)\Rightarrow B\hat{C}A=145°$

iv





5 There are two possible angles $\hat{R}$, corresponding to the possible lengths PR. We apply the sine rule $\frac{\sin(25°)}{5}=\frac{\sin(\hat{R}_1)}{7}\to\hat{R}_1=\sin^{-1}\left(\frac{7}{5}\sin(25°)\right)=36.28°$. The other angle is $\hat{R}_2=180-\hat{R}_1=143.72°$. Here $\hat{Q}_i=180°-25°-\hat{R}_i$, as the internal angles of a triangle sum to $180°$. So, $\hat{Q}_1=118.72°$ and $\hat{Q}_2=11.28°$. We use the sine rule again to get the values of *PR*.

$$\frac{\sin(\hat{R}_i)}{7}=\frac{\sin(\hat{Q}_i)}{PR_i}$$

$$PR_i=7\times\frac{\sin(\hat{Q}_i)}{\sin(\hat{R}_i)}$$

Hence $PR_1=10.4\text{ cm}$ and $PR_2=2.31\text{ cm}$.

6 **a** We notice that $h=\sqrt{2^2-1^2}=\sqrt{3}$

i $\frac{\sqrt{3}}{2}$ **ii** $\frac{1}{2}$ **iii** $\frac{1}{2}$ **iv** $\frac{\sqrt{3}}{2}$

b **i** $\cos(x) = \cos(-x)$ so $\cos(-60°) = \frac{1}{2}$ **ii** $\sin\left(\frac{\pi}{6} \times \frac{180}{\pi}\right) = \sin(30°) = \frac{1}{2}$

iii $\cos\left(-\frac{\pi}{6} \times \frac{180}{\pi}\right) = \cos(-30°) = \frac{\sqrt{3}}{2}$

iv $\sin(-x) = -\sin(x)$ so $\sin\left(-\frac{\pi}{3} \times \frac{180}{\pi}\right) = -\sin(60°) = -\frac{\sqrt{3}}{2}$

7 **a** **i** As $\cos(x) = \frac{1}{3}$, then $\sin^2(x) + \left(\frac{1}{3}\right)^2 = 1 \rightarrow \sin(x) = \sqrt{\frac{8}{9}} = \frac{2}{3}\sqrt{2}$

ii As $\tan(x) = \frac{\sin(x)}{\cos(x)} \Rightarrow \tan(x) = \frac{\frac{2}{3}\sqrt{2}}{\frac{1}{3}} = 2\sqrt{2}$

b **i** $\cos(-x) = \frac{1}{3}$ **ii** $\sin(-x) = -\frac{2\sqrt{2}}{3}$

8 **a** We know $\tan(x) = \frac{\sin(x)}{\cos(x)} = \frac{\sin(x)}{2\sin(x)} = \frac{1}{2}$

b **i** We know $\sin^2(x) + \cos^2(x) = 1 \rightarrow \sin^2(x) + 4\sin^2(x) = 1$. Hence $\sin(x) = \frac{1}{\sqrt{5}}$

ii As $\cos(x) = 2\sin(x)$, then $\cos(x) = \frac{2}{\sqrt{5}}$

c Because in this domain $\sin(x)$ would be negative and $\cos(x)$ positive, so the equality $2\sin(x) = \cos(x)$ would no longer hold.

Exercise 8C

1 **a** $y = 2\sin(3(x-4)) - 5$

i $a = 2$ **ii** $y = -5$ **iii** $P = \frac{2\pi}{3}$ **iv** $c = 4$ to the right

b $y = -3\cos(\pi(x+1)) + 3$

i $a = 3$ **ii** $y = 3$ **iii** $P = \frac{2\pi}{\pi} = 2$ **iv** $c = 1$ to the left

c $y = 4\sin(3x - 6) = 4\sin(3(x-2)) + 0$

i $a = 4$ **ii** $y = 0$ **iii** $P = \frac{2\pi}{3}$ **iv** $c = 2$ to the right

d $y = \cos(2x + 5) - 1 = \cos\left(2\left(x + \frac{5}{2}\right)\right) - 1$

i $a = 1$ **ii** $y = -1$ **iii** $P = \frac{2\pi}{2} = \pi$ **iv** $c = \frac{5}{2}$ to the left

2 **a** $a=\frac{1}{2}(3-(-1))=2$, $d=\frac{1}{2}(3+(-1))=1$, $P=4-0=4\Rightarrow b=\frac{2\pi}{P}=\frac{\pi}{2}$.

Hence $y=2\sin\left(\frac{\pi}{2}x\right)+1$

b $a=\frac{1}{2}(1.5-(-3.5))=2.5$, $d=\frac{1}{2}(1.5+(-3.5))=-1$, $P=6-0=6\rightarrow b=\frac{2\pi}{P}=\frac{\pi}{3}$

Hence $y=2.5\sin\left(\frac{\pi}{3}x\right)-1$

3 **a** **i** $a=\frac{1}{2}(3-(-5))=4$, $d=\frac{1}{2}(3+(-5))=-1$, $P=5.5-1.5=4\Rightarrow b=\frac{2\pi}{P}=\frac{\pi}{2}$,

$c=\frac{-0.5+1.5}{2}=\frac{1}{2}$.

Hence, $y=4\sin\left(\frac{\pi}{2}\left(x-\frac{1}{2}\right)\right)-1$

ii $a=\frac{1}{2}(5-(-1))=3$, $d=\frac{1}{2}(5+(-1))=2$, $P=2(6.5-3.5)=6\Rightarrow b=\frac{2\pi}{P}=\frac{\pi}{3}$,

$c=\frac{0.5+3.5}{2}=2$.

Hence, $y=3\sin\left(\frac{\pi}{3}(x-2)\right)+2$

b **i** As $\sin(x)=\cos\left(x-\frac{\pi}{2}\right)$, then $y_i=4\cos\left(\frac{\pi}{2}\left(x-\frac{1}{2}\right)-\frac{\pi}{2}\right)-1=4\cos\left(\frac{\pi}{2}\left(x-\frac{3}{2}\right)\right)-1$

ii $y_{ii}=3\cos\left(\frac{\pi}{3}(x-2)-\frac{\pi}{2}\right)+2=3\cos\left(\frac{\pi}{3}\left(x-\frac{7}{2}\right)\right)+2$.

c The translation will be one quarter of the period to the left. Hence $k=\frac{1}{4}$

4 **a** Dates are not real numbers, whereas the day numbers are.

b Each time is represented as an hours part (h) and a minutes part (m). Therefore, "hours after midnight" is $h+\frac{m}{60}$.

c Note: some calculators may give slightly different answers

$f(t)=1.5\sin(0.017t+1.67)+6.4$

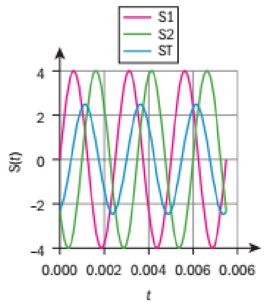
d Notice that 02-Feb-2019 is day 398 in our scale. Hence, $f(398)=7.65=7:39$

e The times of sunrise for each date in subsequent years are nearly the same, so whilst this is extrapolation, it will be reliable for future dates.

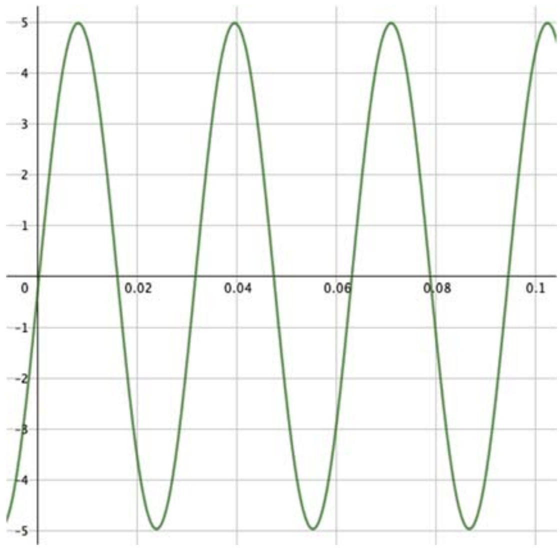
5 **a** We know $b=\frac{2\pi}{P}=2\pi f$, hence, $b=1000\pi=3142$

b $1\text{ms}=0.001\text{s}$, hence, $S_2(t)=4\sin(1000\pi(t-0.001))$

c Looks like a sinewave which is on counterphase with roughly half the amplitude.



6 a



b By inspection we find that max $(0.00825, 4.98)$, min $(0.02396, -4.98)$

c i $a = 4.98$

ii distance between maximim and minimum values is one half of the period so period = $2(0.02396 - 0.00825) = 0.0314$

d $b = \dfrac{2\pi}{0.03142} = 200$ For the delay, one sees the first crossing with the principal axis which, from the GDC is $c \approx 0.00040$. Hence, $S_T(t) = 4.98\sin(200(t - 0.0004)) + 0$

7 a $h_{\text{Max}} = 4.1$, $h_{\text{Min}} = 0.8$. Thus, $a = \frac{1}{2}(4.1 - 0.8) = 1.65$, $d = \frac{1}{2}(4.1 + 0.8) = 2.45$

b We take the mean between the time difference on the highest and lowest tides to get the period: $P = \frac{1}{2}((15.15 - 3.033) + (21.32 - 8.9)) = 12.27$. Hence, $b = \frac{2\pi}{P} = 0.512$

c We know that $\sin(x)$ has a first maximum at $x = \frac{\pi}{2}$. The first maximum tide height happens at $t_{\text{Max}} = 3.033$, so we want that $0.512(t_{\text{Max}} - c) = \frac{\pi}{2} \Rightarrow c = 3.033 - \frac{\pi}{2 \times 0.512}$ $= -0.035$. Hence, the model is $\text{height}(t) = 1.65\sin(0.512(t + 0.035)) + 2.45$

d The regression produces the equation $f(t) = 1.54\sin(0.518(t-0.03)) + 2.5$ which is similar to the model we got by inspection of the table.

e Using the rounded values from the answer to part c we obtain the following table (though answers will differ if more figures are used).

Time, t	Height, h(t)
07:52	1.15 m
10:24	1.12 m
20:08	1.16 m
22.49	1.19 m

The lowest depth of water that is rowable is about 1.12 m.

Exercise 8D

1 a $x = \dfrac{-4 \pm \sqrt{16-8}}{4} = 1 \pm \dfrac{1}{\sqrt{2}} = 1.71 \text{ or } 0.29$

b We factorise as $(x-5)(x+1) = 0$, thus $x = 5$ or $x = -1$

c $x = \dfrac{8 \pm \sqrt{64-80}}{8} = 1 \pm \dfrac{\sqrt{-16}}{8} = 1 \pm \dfrac{1}{2}i$

d $x = \pm\sqrt{-10} = \pm\sqrt{10}\,i$

2 a $2a - 3b = 4 + 2i - 9 + 6i = -5 + 8i$

b $ab = (2+i)(3-2i) = 6 + 2 + 3i - 4i = 8 - i$

c $\dfrac{a}{b} = \dfrac{ab^*}{|b|^2} = \dfrac{1}{3^2+2^2}(2+i)(3+2i) = \dfrac{1}{13}(6 - 2 + 3i + 4i) = \dfrac{1}{13}(4+7i)$

d $b^2 = (3-2i)(3-2i) = 9 - 4 - 12i = 5 - 12i$

e $c^3 = c^2c = (1 - 1 - 2i)(1-i) = -2i(1-i) = -2 - 2i$

f $\dfrac{a^4}{b} = \dfrac{a}{b}a^3$. From **2 c**, $\dfrac{a}{b} = \dfrac{1}{13}(4+7i)$. $a^3 = a^2a = (3+4i)(2+i) = 2 + 11i$.

Hence $\dfrac{a^4}{b} = \dfrac{1}{13}(4+7i)(2+11i) = \dfrac{1}{13}(-69+58i)$

3 Using the quadratic formula $x = \dfrac{4 \pm \sqrt{16-20}}{2} = \dfrac{4 \pm \sqrt{-4}}{2} = 2 \pm i$, hence,

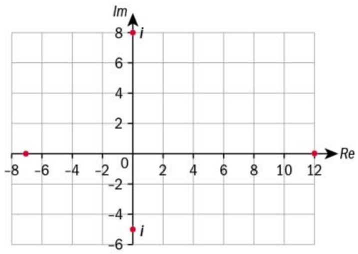
$x^2 - 4x + 5 = (x - (2+i))(x - (2-i))$

4 The other root is $3 + \mathrm{i}$ hence the equation can be written as $(x - (3+\mathrm{i}))(x - (3-\mathrm{i})) = 0$

This expands to give $x^2 - (3+\mathrm{i})x - (3-\mathrm{i})x + (3-\mathrm{i})(3+\mathrm{i}) = x^2 - 6x + 10$

Exercise 8E

1



a $8\text{cis}\left(\frac{\pi}{2}\right)$ **b** $7\text{cis}\pi$ **c** $12\text{cis}0$ **d** $5\text{cis}\left(-\frac{\pi}{2}\right)$

2 **a** $r = \sqrt{13} \approx 3.61,\ \theta = \tan^{-1}\left(\frac{3}{2}\right) = 0.983$ but fourth quadrant so -0.983

b $r = \sqrt{29} \approx 5.39,\ \theta = \tan^{-1}\left(\frac{5}{2}\right) = 1.19$ rad

c $r = \sqrt{10} \approx 3.16,\ \theta = \tan^{-1}\left(\frac{1}{3}\right) = 0.322$ rad but 4th quadrant so -0.322

d $r = \sqrt{20} \approx 4.47,\ \tan^{-1}\left(\frac{1}{2}\right) = 0.464$ rad, but 3rd quadrant so $\theta = 0.464 - \pi = -2.68$ rad

e $r = \sqrt{29} \approx 5.39,\ \tan^{-1}\left(\frac{2}{5}\right) = 0.381$ rad, but 2nd quadrant so $\theta = \pi - 0.381 = 2.76$ rad

f $r = \sqrt{10} \approx 3.16,\ \tan^{-1}(3) = 1.25$ rad but 4$^{\text{th}}$ quadrant so -1.25

3 We use that $\tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}$, $\tan\left(\frac{\pi}{4}\right) = 1$, $\tan\left(\frac{\pi}{3}\right) = \sqrt{3}$.

a $r = \sqrt{4^2 + 4^2} = 4\sqrt{2},\ \theta = \tan^{-1}(1) = \frac{\pi}{4} \to z = 4\sqrt{2}\,\text{cis}\left(\frac{\pi}{4}\right)$

b $r = \sqrt{2^2 + 2^2 \times 3} = \sqrt{16} = 4,\ \theta = \tan^{-1}\left(\sqrt{3}\right) = \frac{\pi}{3} \to z = 4\text{cis}\left(\frac{\pi}{3}\right)$

c $r = \sqrt{3^2 \times 3 + 3^2} = \sqrt{36} = 6,\ \theta = \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) = -\frac{\pi}{6} \to z = 6\text{cis}\left(-\frac{\pi}{6}\right)$

d $r = \sqrt{3+1} = 2,\ \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$, but 3rd quadrant so

$$\theta = \frac{\pi}{6} - \pi = -\frac{5\pi}{6} \text{ rad} \Rightarrow z = 2\ \text{cis}\left(-\frac{5\pi}{6}\right)$$

e $r = \sqrt{5^2 + 5^2} = 5\sqrt{2}$, $\tan^{-1}(-1) = -\frac{\pi}{4}$, but 2nd quadrant so

$\theta = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$ rad $\Rightarrow z = 5\sqrt{2}\ \text{cis}\left(\frac{3\pi}{4}\right)$

f $r = \sqrt{7^2 + 7^2 \times 3} = \sqrt{7^2 \times 4} = 14$, $\theta = \arctan\left(\sqrt{3}\right) = \frac{\pi}{3} \Rightarrow z = 14\ \text{cis}\left(-\frac{\pi}{3}\right)$ as it is in the 4th quadrant

4 **a** $3\text{cis}(60°) = 1.5 + 2.60i$

b $4\text{cis}(120°) = -2 + 3.46i$

c $2\text{cis}(-150°) = -1.73 - i$

d $5\text{cis}(0.4) = 4.61 + 1.95i$

e $2.4\text{cis}(1.9) = -0.78 + 2.27i$

f $3.8\text{cis}(-0.6) = 3.14 - 2.15i$

Exercise 8F

1 We will use $a = 1 + \sqrt{3}i$, $b = -1 + i$, $c = \sqrt{3} - i$.

a $|a| = \sqrt{1+3} = 2$, $\theta = \tan^{-1}\left(\sqrt{3}\right) = \frac{\pi}{3} \Rightarrow a = 2\ \text{cis}\left(\frac{\pi}{3}\right)$

$|b| = \sqrt{1+1} = \sqrt{2}$, $\tan^{-1}(1) = \frac{\pi}{4}$, but 2nd quadrant so $\theta = \pi - \frac{\pi}{4} = \frac{3\pi}{4} \Rightarrow b = \sqrt{2}\ \text{cis}\left(\frac{3\pi}{4}\right)$

$|c| = \sqrt{3+1} = 2$, $\theta = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6} \Rightarrow c = 2\ \text{cis}\left(-\frac{\pi}{6}\right)$ as it is in the 4th quadrant

b **i** $ac = 4e^{i(\pi/3 - \pi/6)} = 4e^{i\pi/6}$, so $x = 4 \times \sqrt{3}/2 = 2\sqrt{3}$ and $y = 4 \times 1/2 = 2$.

$ac = 2\sqrt{3} + 2i$

ii $\frac{a}{c} = \frac{2e^{i\pi/3}}{2e^{-i\pi/6}} = e^{i(\pi/3 + \pi/6)} = e^{i\pi/2} = i$

iii $b^4 = \sqrt{2}^4 \text{cis}\left(4 \times \frac{3\pi}{4}\right) = 4\text{cis}(3\pi) = 4\text{cis}(\pi) = -4$

iv $a^3 = 2^3\text{cis}\left(3 \times \frac{\pi}{3}\right) = 8\text{cis}(\pi) = -8$, $b^2 = \sqrt{2}^2\text{cis}\left(2 \times \frac{3\pi}{4}\right) = 2\text{cis}\left(\frac{3\pi}{2}\right) = -2i$. Thus,

$\frac{a^3}{b^2} = \frac{-8}{-2i} = -4i$

v $\frac{a^2 b}{c^2} = \left(\frac{a}{c}\right)^2 b = i^2 \times b = -b = 1 - i$

2 We first find the argument of z $\arg(z) = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$ or $30°$

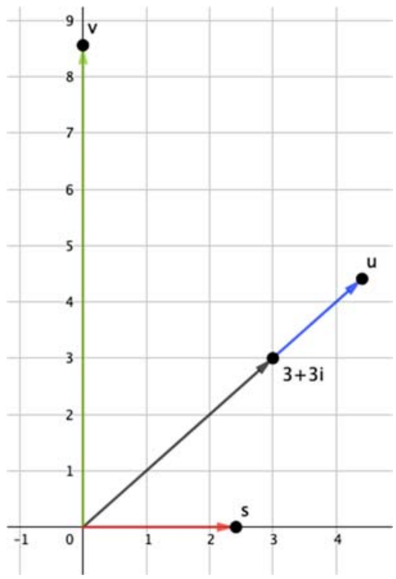
Thus $z^n = 2^n\left(\cos\left(n\dfrac{\pi}{6}\right) + i\sin\left(n\dfrac{\pi}{6}\right)\right)$. This will have imaginary part 0 when

$\sin\left(n\dfrac{\pi}{6}\right) = 0 \Rightarrow n\dfrac{\pi}{6} = k\pi$, where k is an integer. Hence $n = 6k$ where $k \in$, i.e.

$n = \{\ldots -6, 0, 6, 12, 18, 24\ldots\}$

3 **a** $r = \sqrt{2+2} = 2,\ \theta = \text{atan}(1) = \dfrac{\pi}{4} \rightarrow z = 2e^{\frac{i\pi}{4}}$

b $3+3i$ can be plotted as the coordinate $(3,3)$ in the Argand diagram.



c Points plotted on the Argand diagram in part **b**

d u is a translation of $\begin{pmatrix}\sqrt{2}\\ \sqrt{2}\end{pmatrix}$.

v is a rotation of $\dfrac{\pi}{4}$ anticlockwise and an enlargement by a factor of 2.

s is a rotation of $\dfrac{\pi}{4}$ clockwise and an enlargement by a factor of 0.5.

4 If $z = x + iy$, then $z^* = x - iy$. Thus, $|z^*| = |z| = r$ and

$\arg(z^*) = \tan^{-1}\left(-\dfrac{y}{x}\right) = -\tan^{-1}\left(\dfrac{y}{x}\right) = -\arg(z) = -\theta$. Hence $z^* = re^{-i\theta}$.

$zz^* = r^2e^{i\theta - i\theta} = r^2$ and therefore $\text{Im}(zz^*) = 0$

5 $e^{i\pi} = \cos(\pi) + i\sin(\pi) = -1 \rightarrow e^{i\pi} + 1 = 0$.

6 **a** Recall that $\tan\left(\dfrac{\pi}{4}\right) = 1 \rightarrow \sin\left(\dfrac{\pi}{4}\right) = \cos\left(\dfrac{\pi}{4}\right)$. So, the line described by $\arg(z) = \dfrac{\pi}{4}$ is the line where $Re(z) = Im(z)$, which is the line with slope 1 in the Argand diagram.

b We know $z(t) = 0.5te^{\frac{\pi}{2}ti}$

i Crossing the imaginary axis happens when $\text{Re}(z) = 0 \to \cos(\theta) = 0 \to \theta = \frac{\pi}{2}$. Hence

$\frac{\pi}{2}t = \frac{\pi}{2} \to t = 1$.

ii Crossing the real axis happens when $\text{Im}(z) = 0 \to \sin(\theta) = 0 \to \theta = \pi$.

Hence $\frac{\pi}{2}t = \pi \to t = 2$.

c Imaginary crossing: $z_I = 0 + 0.5i$, real crossing: $z_R = 1 + 0i$.

d We know that all z_1, z_2, z_3 have argument θ. Thus, the intersection points will happen when

$0.5te^{\frac{\pi}{2}ti} = r_1e^{i\theta}$, $r_2e^{i(\theta+2\pi)}$ and $r_3e^{i(\theta+4\pi)}$

The argument for A will be θ so $\frac{\pi}{2}t = \theta \Rightarrow t = \frac{2\theta}{\pi}$ and the value of A is

$$0.5 \times \frac{2\theta}{\pi}e^{\theta i} = \frac{\theta}{\pi}e^{i\theta}$$

The curve will next intersect the line when $\frac{\pi}{2}t = 2\pi + \theta \Rightarrow t = \frac{2(2\pi+\theta)}{\pi} = 4 + \frac{2\theta}{\pi}$

Hence the value of B is $\left(2 + \frac{\theta}{\pi}\right)e^{(2\pi+\theta)i} = \left(2 + \frac{\theta}{\pi}\right)e^{\theta i}$

The curve will next intersect the line when $\frac{\pi}{2}t = 4\pi + \theta \Rightarrow t = \frac{2(4\pi+\theta)}{\pi} = 8 + \frac{2\theta}{\pi}$

Hence the value of B is $\left(4 + \frac{\theta}{\pi}\right)e^{(4\pi+\theta)i} = \left(4 + \frac{\theta}{\pi}\right)e^{\theta i}$

e $|z_2| - |z_1| = \left(2 + \frac{\theta}{\pi}\right) - \left(\frac{\theta}{\pi}\right) = 2$, $|z_3| - |z_2| = \left(4 + \frac{\theta}{\pi}\right) - \left(2 + \frac{\theta}{\pi}\right) = 2$.

Since $|z_{n+1}| - |z_n| = \left(2(n+1) + \frac{\theta}{\pi}\right) - \left(2n + \frac{\theta}{\pi}\right) = 2$, this pattern will continue.

Exercise 8G

1 Let $V_1 = 110\sin(ax)$ and $V_2 = 110\sin(ax + 60°)$. We know the maximum output of a sine wave is just its modulus, so

$\max(V_T) = |V_1 + V_2| = |110(\text{cis}(0) + \text{cis}(60°))| = 110\left|1 + \frac{1}{2} + i\frac{\sqrt{3}}{2}\right| = 110\sqrt{\frac{9}{4} + \frac{3}{4}} = 110\sqrt{3}$ V.

2 **a** Let $V_1 = 110\sin(at+0) = 110\,Im(\text{cis}(at+0)) = 110\,Im(\text{cis}(at)\text{cis}(0))$. Similarly, we define $V_2 = 110\,Im\left(\text{cis}(at)\text{cis}\left(\frac{2\pi}{3}\right)\right)$ and $V_3 = 110\,Im\left(\text{cis}(at)\text{cis}\left(\frac{4\pi}{3}\right)\right)$, which are separated with a phase of $\frac{2\pi}{3}$ each.

Thus,

$$V_1 + V_2 + V_3 = 110\,Im\left(\text{cis}(at)\left(\text{cis}(0) + \text{cis}\left(\frac{2\pi}{3}\right) + \text{cis}\left(\frac{4\pi}{3}\right)\right)\right)$$
$$= 110\,Im\left(\text{cis}(at)\left(1 - 0.5 + i\frac{\sqrt{3}}{2} - 0.5 - \frac{i\sqrt{3}}{2}\right)\right) \quad .$$
$$= 110\,Im(\text{cis}(at)(0 - 0i)) = 0$$

b Connecting in reverse is equivalent to reflecting in the x-axis as all positive becomes negative and vice versa. That is the same as shifting by 180°.

c In the shifted case

$$-V_1 + V_2 + V_3 = 110\,Im\left(\text{cis}(at)\left(-1 - 0.5 + i\frac{\sqrt{3}}{2} - 0.5 - \frac{i\sqrt{3}}{2}\right)\right) = 220\,Im(\text{cis}(at)) = 220\sin(at)$$

3 Given $V_1 = 6\sin(at)$ and $V_2 = 10\sin(at + 40°)$, the combined maximum output is $|V_1 + V_2| = |6\,cis(0) + 10\,\text{cis}(40°)| = |6 + 7.66 + i6.43| = 15.1\,\text{V}$.

4 $r = \left|3\text{cis}\left(\frac{\pi}{12}\right) + 4\text{cis}\left(\frac{3\pi}{4}\right)\right| = |2.90 + 0.78i - 2.83 + 2.83i| = |0.07 + 3.61i| = 3.61$ and

$\alpha = \arg\left(3\text{cis}\left(\frac{\pi}{12}\right) + 4\text{cis}\left(\frac{3\pi}{4}\right)\right) = \arg(0.07 + 3.61i) = \tan^{-1}\left(\frac{3.61}{0.07}\right) = 1.55$. Thus,

$f(x) + g(x) = 3.61\sin(3x + 1.55)$

5 **a** The length of the day is simply $h(t) = g(t) - f(t)$,

$$\begin{aligned} h(t) &= g(t) - f(t) \\ &= 2.19\sin(0.0165t - 1.23) - 2.14\sin(0.0165t + 1.81) + 18 - 5.97 \\ &= 2.19\sin(0.0165t - 1.23) - 2.14\sin(0.0165t + 1.81) + 12.03 \end{aligned}$$

$$\begin{aligned} h(t) &= \text{Im}\left(2.19e^{(0.0165t-1.23)i} - 2.14e^{(0.0165t+1.81)i}\right) + 12.03 \\ &= \text{Im}\left(e^{0.0165ti}\left(2.19e^{-1.23i} - 2.14e^{1.81i}\right)\right) + 12.03 \\ &= \text{Im}\left(e^{0.0165ti} \times 4.32e^{-1.28i}\right) + 12.03 \\ &= 4.32\sin(0.0165t - 1.28) + 12.03 \end{aligned}$$

b Longest day is $4.32 + 12.03 = 16.35$ hours

Shortest day is $-4.32 + 12.03 = 7.71$ hours

c From a graph on the GDC or by solving the equation $\sin(0.0165t - 1.28) = 1$ the longest day will be when $t = 172.8$, which is 21 June, while the shortest one will be when $\sin(0.0165t + \alpha) = -1 \Rightarrow 0.0165t - 1.28 = \frac{3\pi}{2}$, hence $t = 363.2$, which is 29 December.

Chapter review

1 $A_{\text{shaded}} = A_{\text{circle}} - A_{\text{not shaded}}$, where $A_{\text{notshaded}} = \frac{1}{2}A_{\text{circle}} + A_{\text{segment}_{AB}}$, where

$A_{\text{segment}_{AB}} = A_{\text{sector}_{APB}} - A_{\text{triangle}_{APB}}$.

Let's get all of this in terms of R:

$A_{\text{triangle}_{APB}} = \frac{1}{2}(b \times h) = \frac{1}{2}(AB + R) = \frac{1}{2}(2R \times R) = R^2$

$A_{\text{sector}_{APB}} = \frac{1}{2}AP^2 \times \frac{\pi}{2}$, where $AP^2 = OP^2 + OA^2 = 2R^2$, using Pythagoras theorem. Thus,

$A_{\text{sector}_{APB}} = \frac{\pi}{2}R^2$

Hence, $A_{\text{segment}_{AB}} = \frac{\pi}{2}R^2 - R^2 = \left(\frac{\pi}{2} - 1\right)R^2$

Now, $A_{\text{circle}} = \pi R^2$

Hence, $A_{\text{not shaded}} = \frac{\pi}{2}R^2 + \left(\frac{\pi}{2} - 1\right)R^2 = (\pi - 1)R^2$

Finally, $A_{\text{shaded}} = \pi R^2 - (\pi - 1)R^2 = R^2$.

2 **a** Area $= \frac{1}{2}bc\sin A = \frac{1}{2} \times 1 \times 2 \times \sin\left(\frac{2\pi}{3}\right) = 0.866$

b $BC^2 = 2^2 + 1^2 - 2 \times 2 \times 1 \times \cos\left(\frac{2\pi}{3}\right) = 7 \Rightarrow BC = 2.65$

3 The perimeter is the sum of the three sides, so $P = 5 + 5 + 5 \times 2 = 20\text{ cm}$. The area is just the arc sector $A = \frac{1}{2}5^2 \times 2 = 25\text{ cm}^2$.

4 $\sqrt{3} + i = r\,e^{i\theta}$, where $r = \sqrt{3+1} = 2$ and $\theta = \arctan\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$. Thus, $\left(2\text{cis}\left(\frac{\pi}{6}\right)\right)^{10} = 2^{10}\text{cis}\left(\frac{10\pi}{6}\right)$,

so $\arg\left(\left(\sqrt{3} + i\right)^{10}\right) = \frac{5\pi}{3}$ or $-\frac{\pi}{3}$

5 $|z_1| = \sqrt{1+1} = \sqrt{2}$, $\arg(z_1) = \tan^{-1}(1) = \frac{\pi}{4}$. Hence $z_1 = \sqrt{2}\,\text{cis}\left(\frac{\pi}{4}\right)$

$|z_2| = \sqrt{3+1} = 2$, $\arg(z_2) = \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) = -\frac{\pi}{6}$. Hence $z_2 = 2\text{cis}\left(-\frac{\pi}{6}\right)$

a $z_1 z_2 = 2\sqrt{2}\text{cis}\left(\frac{\pi}{4} + \frac{\pi}{6}\right) = 2\sqrt{2}\,\text{cis}\left(\frac{\pi}{12}\right)$

b $\frac{z_1}{z_2} = \frac{\sqrt{2}}{2}\text{cis}\left(\frac{\pi}{4} - \frac{\pi}{6}\right) = \frac{1}{\sqrt{2}}\text{cis}\left(\frac{5\pi}{12}\right)$

c $\frac{z_1^3 z_2^3}{i} = -i(z_1 z_2)^3 = -i\left(2\sqrt{2}\,\text{cis}\left(\frac{5\pi}{12}\right)\right)^3 = \text{cis}\left(-\frac{\pi}{2}\right)\left(16\sqrt{2}\text{cis}\left(\frac{5\pi}{4}\right)\right) = 16\sqrt{2}\text{cis}\left(\frac{3\pi}{4}\right)$

6 First, we construct a director vector $v = B - A = -2 - 4i$. Then, we multiply it by i to get a rotation of $\frac{\pi}{2}$ of this vector. Thus, $v_\perp = i(-2 - 4i) = 4 - 2i$.

Finally, $C = B + v_\perp = 1 - 2i$, $D = A + v_\perp = 3 + 2i$

7 a $|w| = \sqrt{4+4} = \sqrt{8}$ and $\arg(w) = \tan^{-1}(1) = \frac{\pi}{4}$. Hence, $w = \sqrt{8}\,\text{cis}\left(\frac{\pi}{4}\right)$.

b $w^4 = \sqrt{8}^4\text{cis}\left(4 \times \frac{\pi}{4}\right) = -64$, $z^6 = \text{cis}\left(6 \times \frac{5\pi}{6}\right) = \text{cis}(5\pi) = -1$

Hence, $w^4 z^6 = 64$

8 We have $|z_1||z_2| = 2, \frac{|z_1|}{|z_2|} = 2 \rightarrow 2|z_2|^2 = 2$, so $|z_2| = 1$, $|z_1| = 2$

$\theta_1 + \theta_2 = \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) = -\frac{\pi}{6}$, $\theta_1 - \theta_2 = \frac{\pi}{2} \rightarrow \theta_1 = \frac{\pi}{2} + \theta_2$, so $\theta_1 = \frac{\pi}{6}$, $\theta_2 = -\frac{\pi}{3}$

9 $A_f = |2\text{cis}(60°) + 3\text{cis}(30°)| = \left|2\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) + 3\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)\right| = \left|\left(1 + \frac{3\sqrt{3}}{2}\right) + \left(\sqrt{3} + \frac{3}{2}\right)i\right| = 4.84$

10 $I_T = |20\text{cis}(0°) + 10\text{cis}(60°)| = \left|20 + 10\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\right| = |25 + 5\sqrt{3}i| = \sqrt{5^4 + 5^2 \times 3} = 5\sqrt{28} = 26.5$ amps .

If connected in reverse, $I_T = \left|20 - 10\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\right| = |15 - 5\sqrt{3}i| = 10\sqrt{3} = 17.3$ amps

Exam style questions

11 a The principal axis is $\frac{5.5 + 1.5}{2}$

Hence $p = 3.5$

The amplitude is $\frac{5.5 - 1.5}{2} = 2$

Hence $q = 2$

The period is 120

$120 = \frac{360}{r}$

Hence $r = 3$

So $y = 3.5 + 2\cos 3x$

b $17.2 < x < 62.6$

and $137.2 < x \leq 180$

12 Area $OAD = \frac{1}{2}r^2\theta$

$= \frac{1}{2} \times 10^2 \times \frac{6}{5}$

$= 60$ cm^2.

Area $OBC = 2 \times 60 = 120$ cm^2

$\frac{1}{2} \times R^2 \times \frac{6}{5} = 120$

$\Rightarrow R = \sqrt{200} = 10\sqrt{2}$

Perimeter

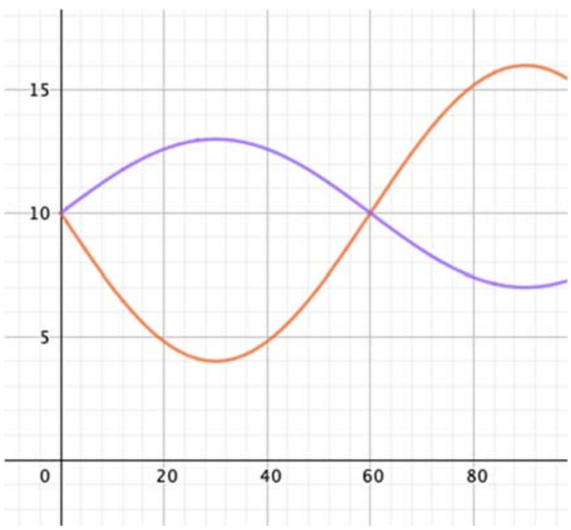
$= BC + AD + 2 \times BA$

$= 10\sqrt{2} \times \frac{6}{5} + 10 \times \frac{6}{5} + 2\left(10\sqrt{2} - 10\right)$

$= 12\sqrt{2} + 12 + 20\sqrt{2} - 20$

$= 32\sqrt{2} - 8$ cm.

13 a



b The principal axis is $\frac{16+4}{2} = (10)$

Hence $p = 10$

The amplitude is $\frac{16-4}{2}(= 6)$

Hence $q = 6$

The period is $2 \times 60 = 120$

$120 = \frac{360}{r}$

Hence $r = 3$

So $y = 10 - 6\sin 3x$

14 a $|100\,\text{cis}(0) + 180\,\text{cis}(90)| = |100 + 180i| = \sqrt{100^2 + 180^2} = 206\text{V}$

b

$\arg(100 + 180i) = \tan^{-1}\left(\frac{180}{100}\right) = 60.9°$

$45 + 60.9 = 105.9°$

15 a $z_1 z_2 = 4\,\text{cis}\left(-\frac{\pi}{3}\right)3\,\text{cis}\left(\frac{5\pi}{6}\right) = 12\,\text{cis}\left(\frac{5\pi}{6} - \frac{\pi}{3}\right)$

$= 12\,\text{cis}\left(\frac{\pi}{2}\right)$

$= 0 + 12\text{i}$

b $\frac{z_1}{z_2} = \frac{4\,\text{cis}\left(-\frac{\pi}{3}\right)}{3\,\text{cis}\left(\frac{5\pi}{6}\right)} = \frac{4}{3}\text{cis}\left(-\frac{\pi}{3} - \frac{5\pi}{6}\right)$

$= \frac{4}{3}\text{cis}\left(-\frac{7\pi}{6}\right) = \frac{4}{3}\text{cis}\left(\frac{5\pi}{6}\right)$

So $\left(\frac{z_1}{z_2}\right)^3 = \left(\frac{4}{3}\text{cis}\left(\frac{5\pi}{6}\right)\right)^3 = \frac{64}{27}\text{cis}\left(\frac{15\pi}{6}\right)$

$= \frac{64}{27}\text{cis}\left(\frac{\pi}{2}\right)$

$= 0 + \frac{64}{27}\text{i}$

c $z_1{}^2 = 16\,\text{cis}\left(-\frac{2\pi}{3}\right)$

So $\left(z_1{}^2\right)^* = 16\,\text{cis}\left(\frac{2\pi}{3}\right)$

$= 16\left(\cos\frac{2\pi}{3} + \text{i}\sin\frac{2\pi}{3}\right)$

$= 16\left(-\frac{1}{2} + \text{i}\frac{\sqrt{3}}{2}\right)$

$= -8 + 8\sqrt{3}\text{i}$

16 $f(x) + g(x) = \text{Im}\left(e^{\left(2x+\frac{\pi}{3}\right)i} + 2e^{\left(2x+\frac{\pi}{4}\right)i}\right) = \text{Im}\left(e^{2xi}\left(e^{\frac{\pi}{3}i} + 2e^{\frac{\pi}{4}i}\right)\right)$

$\mathrm{Im}\left(e^{2xi} \times 2.977e^{0.872i}\right) = 2.977\sin\left(2x + 0.872\right)$

Hence $r = 2.98$, $\alpha = 0.872$

9 Modelling with matrices: storing and analysing data

Skills check

1 $u_n = 18 \times \left(-\frac{2}{3}\right)^{n-1}$.

2 $S_{15} = 18 \times \dfrac{1 - \left(-\frac{2}{3}\right)^{15}}{1 - \frac{2}{3}} = 10.8$

3 $S_n = 6 \times \dfrac{1 - \left(\frac{1}{2}\right)^{n}}{1 - \frac{1}{2}}, S_{12} = 12.0$.

Exercise 9A

1 $a_{1,2} = -3,\ \ a_{2,3} = 6,\ \ b_{3,1} = 5,\ \ c_{2,2} = 0,\ \ c_{3,1} = 1$.

2 **a** 3×4

b $q_{2,4} = 100$, the amount of product 2 produced by manufacturer 4

c $\boldsymbol{T} = \begin{pmatrix} 960 \\ 1200 \\ 990 \\ 335 \end{pmatrix}$.

3 **a** $3\boldsymbol{W} = \begin{pmatrix} 3 & -6 & 9 \\ 9 & 6 & -3 \end{pmatrix}$

b Not possible because the dimensions of the matrices don't agree.

c $4\boldsymbol{U} + \boldsymbol{S} = \begin{pmatrix} 35 & -14 & -7 \\ 28 & 29 & 1 \\ -9 & 10 & -2 \end{pmatrix}$

d $\frac{1}{3}\boldsymbol{T} - \frac{1}{2}\boldsymbol{V} = \begin{pmatrix} 0 & -3 \\ 4 & 8 \end{pmatrix}$.

4 **a** $k = 0.075, \boldsymbol{T} = \begin{pmatrix} 2433.75 \\ 1404.38 \\ 1813.13, \\ 1443.75 \end{pmatrix}$.

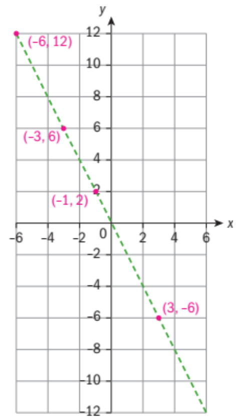
b $\boldsymbol{C} = P + T = (1 + \boldsymbol{k})\boldsymbol{P} = 1.075\boldsymbol{P}$.

5 **a** **i** $2\boldsymbol{X} = \begin{pmatrix} -6 \\ 12 \end{pmatrix}$

ii $\frac{1}{3}\boldsymbol{X} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$

iii $-\boldsymbol{X} = \begin{pmatrix} 3 \\ -6 \end{pmatrix}$.

b



c The set of points described by kX is a line going through the origin and point, P, i.e. $y=-2x$.

6

$$\mu\begin{pmatrix}-9 & 12\\ -6 & 3\end{pmatrix}+\lambda\begin{pmatrix}1 & -2\\ 3 & -1\end{pmatrix}=\begin{pmatrix}-8 & 12\\ -10 & 4\end{pmatrix}$$

Hence,

$$\begin{aligned}-9\mu+\lambda&=-8\\ 12\mu-2\lambda&=12\\ -6\mu+3\lambda&=-10\\ 3\mu-\lambda&=4\end{aligned}$$

Solving, $\lambda=-2, \mu=\frac{2}{3}$.

Exercise 9B

1 **a** $(\boldsymbol{C}+\boldsymbol{F})\boldsymbol{A}=\begin{pmatrix}-2 & -1\\ 1 & 6\\ 3 & 1\end{pmatrix}\times\begin{pmatrix}1 & -2\\ 2 & 3\end{pmatrix}=\begin{pmatrix}-4 & 1\\ 13 & 16\\ 5 & -3\end{pmatrix}$.

b $\boldsymbol{DE}=\begin{pmatrix}3\\ 18\\ -7\end{pmatrix}$.

c $\boldsymbol{ABC}=\begin{pmatrix}4 & -4 & -5\\ 1 & -8 & 11\end{pmatrix}\times\begin{pmatrix}1 & 0\\ -1 & 2\\ -2 & 3\end{pmatrix}=\begin{pmatrix}18 & -23\\ -13 & 17\end{pmatrix}$.

d Impossible because the number of columns of C is not equal to the number of rows of F.

e

$$2\begin{pmatrix}1 & -2\\ 2 & 3\end{pmatrix}-3\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}=\begin{pmatrix}2 & -4\\ 4 & 6\end{pmatrix}-\begin{pmatrix}3 & 0\\ 0 & 3\end{pmatrix}=\begin{pmatrix}-1 & -4\\ 4 & 3\end{pmatrix}$$

2 $m = 2, n = 3$.

3 $B = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

4 a $B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

b $C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$,

c Matrix A is not a square matrix, so its left and right multiplicative identities are not equal, i.e. a multiplicative identity does not exist.

5 a $R = \begin{pmatrix} 0.4 \\ 0.8 \\ 0.6 \end{pmatrix}, N = AR = \begin{pmatrix} 45.0 \\ 36.8 \\ 94.8 \end{pmatrix}$

b $AC = \begin{pmatrix} 410 & 365 & 470 & 430 \\ 348 & 308 & 400 & 360 \\ 890 & 792 & 1020 & 928 \end{pmatrix}$, entry $(AC)_{i,j}$ gives the total daily cost of removing all pollutants while producing the product i at the plant j.

Exercise 9C

1 a Not possible because the number of the columns of matrix A is not equal to the number of the rows of matrix B .

b $BA = \begin{pmatrix} -9 & 4 & 12 \\ 4 & -1 & -4 \end{pmatrix}$

c Not possible because A is not a square matrix.

d $B^2 = \begin{pmatrix} 11 & -8 \\ -4 & 3 \end{pmatrix}$

2 a $SR = \begin{pmatrix} -11 & -13 \\ 18 & -9 \\ 5 & 50 \end{pmatrix}$

b Not possible because the number of the columns of matrix T is not equal to the number of the rows of matrix R,

c

$$\begin{aligned} WR + X &= \begin{pmatrix} -21 & -3 \\ 11 & 7 \end{pmatrix} + \begin{pmatrix} 7 & -2 \\ -6 & 4 \end{pmatrix} \\ &= \begin{pmatrix} -14 & -5 \\ 5 & 11 \end{pmatrix} \end{aligned}$$

d

$$W(S-U)=\begin{pmatrix}1 & -2 & 3\\ 3 & 2 & -1\end{pmatrix}\begin{pmatrix}-1 & 1 & 3\\ -1 & 5 & -5\\ -7 & 4 & 3\end{pmatrix}$$

$$=\begin{pmatrix}-20 & 3 & 22\\ 2 & 9 & -4\end{pmatrix}$$

e Not possible because the number of the columns of matrix $S-U$ is not equal to the number of the rows of matrix W.

3 $(U+I)R=UR+IR=UR+R$

$$UR+R=\begin{pmatrix}1 & -33\\ -22 & -25\\ 8 & 3\end{pmatrix}+\begin{pmatrix}0 & -1\\ 3 & 7\\ -5 & 4\end{pmatrix}=\begin{pmatrix}1 & -34\\ -19 & -18\\ 3 & 7\end{pmatrix},\ (U+I_3)R=\begin{pmatrix}5 & -3 & -2\\ 5 & -3 & 2\\ 0 & 1 & 0\end{pmatrix}\times\begin{pmatrix}0 & -1\\ 3 & 7\\ -5 & 4\end{pmatrix}=\begin{pmatrix}1 & -34\\ -19 & -18\\ 3 & 7\end{pmatrix}.$$

4 $(T+X)(T-X)=T^2-TX+XT-X^2$ and $(T+X)^2=(T+X)(T+X)=T^2+TX+XT+X^2$.

Because matrix multiplication is not commutative then $-TX+XT\neq 0$, so $(T+X)(T-X)=T^2-TX+XT-X^2\neq T^2-X^2$.

Similarly, $TX+XT\neq 2TX$, so $(T+X)^2=T^2+TX+XT+X^2\neq T^2+2TX+X^2$

Either using the GDC or by hand show that

$$(T+X)(T-X)=\begin{pmatrix}-48 & 32\\ 77 & -35\end{pmatrix},\text{but } T^2-X^2=\begin{pmatrix}-62 & 30\\ 62 & -21\end{pmatrix}$$

$$(T+X)^2=\begin{pmatrix}64 & 0\\ -105 & 49\end{pmatrix}\text{ but } T^2+2TX+X^2=\begin{pmatrix}50 & -2\\ -120 & 63\end{pmatrix}$$

5 $$(SU)^2=\begin{pmatrix}2 & 0 & -11\\ 21 & -19 & -3\\ -3 & 3 & 22\end{pmatrix}^2=\begin{pmatrix}37 & -33 & -264\\ -348 & 352 & -240\\ -9 & 9 & 508\end{pmatrix},$$

$$S^2U^2=\begin{pmatrix}-6 & -3 & 11\\ 37 & -22 & -5\\ -15 & 29 & -18\end{pmatrix}\times\begin{pmatrix}1 & -2 & -12\\ 0 & 3 & -20\\ 5 & -5 & 3\end{pmatrix}=\begin{pmatrix}49 & -52 & 165\\ 12 & -115 & -19\\ -105 & 207 & -454\end{pmatrix}.$$

$(SU)^2=(SU)\times(SU)=SUSU\neq SSUU=S^2U^2$ because matrix multiplication is not commutative.

6 $$2(RT)=2\begin{pmatrix}1 & -3\\ -4 & 27\\ -9 & 2\end{pmatrix}=\begin{pmatrix}2 & -6\\ -8 & 54\\ -18 & 4\end{pmatrix},(2R)T=\begin{pmatrix}0 & -2\\ 6 & 14\\ -10 & 8\end{pmatrix}\begin{pmatrix}1 & 2\\ -1 & 3\end{pmatrix}=\begin{pmatrix}2 & -6\\ -8 & 54\\ -18 & 4\end{pmatrix},$$

$$R(2T)=\begin{pmatrix}0 & -1\\ 3 & 7\\ -5 & 4\end{pmatrix}\begin{pmatrix}2 & 4\\ -2 & 6\end{pmatrix}=\begin{pmatrix}2 & -6\\ -8 & 54\\ -18 & 4\end{pmatrix}.$$

Scalar multiplication is commutative with matrices.

7 **a** $AB=BA=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$

b $(2AB)^{10}=2^{10}I_2=\begin{pmatrix}1024 & 0\\ 0 & 1024\end{pmatrix}$

8 **a** The conjecture is not true. Consider, for example, matrices $\boldsymbol{A}=\begin{pmatrix}-3 & 5\\ 1 & -2\end{pmatrix}$, $\boldsymbol{B}=\begin{pmatrix}1 & 2\\ -1 & 3\end{pmatrix}$ and $n=2$. Then, $(\boldsymbol{AB})^2=\begin{pmatrix}-8 & 9\\ 3 & -4\end{pmatrix}^2=\begin{pmatrix}91 & -108\\ -36 & 43\end{pmatrix}$, but $\boldsymbol{A}^2\boldsymbol{B}^2=\begin{pmatrix}14 & -25\\ -5 & 9\end{pmatrix}\begin{pmatrix}-1 & 8\\ -4 & 7\end{pmatrix}=\begin{pmatrix}86 & -63\\ -31 & 23\end{pmatrix}$.

b The conjecture is true because of the scalar commutativity: $(kA)^n=(kA)(kA)\ldots=k^nA^n$

Exercise 9D

1 **a** $A^{-1}=\frac{1}{2}\begin{pmatrix}-5 & 4\\ -3 & 2\end{pmatrix}=\begin{pmatrix}-2.5 & 2.0\\ -1.5 & 1.0\end{pmatrix}$; $AA^{-1}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$

b $A^{-1}=\frac{1}{-3}\begin{pmatrix}0 & 3\\ 1 & -2\end{pmatrix}=\begin{pmatrix}0 & -1\\ -\frac{1}{3} & \frac{2}{3}\end{pmatrix}$; $AA^{-1}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$

c $A^{-1}=\frac{16}{5}\begin{pmatrix}\frac{3}{4} & -\frac{1}{2}\\ -\frac{1}{2} & \frac{3}{4}\end{pmatrix}=\begin{pmatrix}2.4 & -1.6\\ -1.6 & 2.4\end{pmatrix}$; $AA^{-1}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$

d $A^{-1}=\frac{20}{-11}\begin{pmatrix}0.2 & -0.8\\ -0.75 & 0.25\end{pmatrix}=\begin{pmatrix}-\frac{4}{11} & \frac{16}{11}\\ \frac{15}{11} & -\frac{5}{11}\end{pmatrix}$; $AA^{-1}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$

2 **a** $A^{-1}=\begin{pmatrix}-2 & 1 & -4\\ 1 & 0 & 2\\ 1.5 & -0.5 & 2.5\end{pmatrix}$; $AA^{-1}=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$

b $A^{-1}=\begin{pmatrix}\frac{-20}{79} & -\frac{5}{158} & \frac{70}{79}\\ \frac{-52}{79} & \frac{33}{79} & \frac{24}{79}\\ -\frac{9}{79} & \frac{57}{158} & \frac{-8}{79}\end{pmatrix}$; $AA^{-1}=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$

c $A^{-1}=\begin{pmatrix}\frac{24}{175} & \frac{54}{175} & \frac{6}{25}\\ -\frac{36}{25} & -\frac{6}{25} & \frac{12}{25}\\ -\frac{12}{35} & -\frac{27}{35} & \frac{2}{5}\end{pmatrix}$; $AA^{-1}=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$

3 **a** $\boldsymbol{A}=\begin{pmatrix}1 & 1 & 3\\ 3 & 2 & -1\\ 2 & 1 & 1\end{pmatrix}$, $\boldsymbol{X}=\begin{pmatrix}x\\ y\\ z\end{pmatrix}$, $\boldsymbol{B}=\begin{pmatrix}30\\ 20\\ 10\end{pmatrix}$, $\boldsymbol{X}=\boldsymbol{A}^{-1}\boldsymbol{B}=\begin{pmatrix}-0.6 & -0.4 & 1.4\\ 1 & 1 & -2\\ 0.2 & -0.2 & 0.2\end{pmatrix}\begin{pmatrix}30\\ 20\\ 10\end{pmatrix}=\begin{pmatrix}-12\\ 30\\ 4\end{pmatrix}$.

$x=-12, y=30, z=4$

b $A = \begin{pmatrix} 5 & -7 \\ -2 & 5 \end{pmatrix}, X = \begin{pmatrix} x \\ y \end{pmatrix}, B = \begin{pmatrix} 12 \\ 20 \end{pmatrix}, X = A^{-1}B = \frac{1}{11}\begin{pmatrix} 5 & 7 \\ 2 & 5 \end{pmatrix}\begin{pmatrix} 12 \\ 20 \end{pmatrix} = \begin{pmatrix} \frac{200}{11} \\ \frac{124}{11} \end{pmatrix}$

$x = 200/11, y = 124/11$

c

$$A = \begin{pmatrix} 2 & 1 & -1 & 4 \\ 0 & 5 & -3 & 0 \\ 4 & -3 & 0 & 1 \\ 1 & 0 & 2 & -8 \end{pmatrix}, X = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}, B = \begin{pmatrix} -2 \\ 4 \\ 1.75 \\ -0.5 \end{pmatrix},$$

$$X = A^{-1}B = \begin{pmatrix} 0.2440945 & 0.0078740 & 0.0944882 & 0.1338583 \\ 0.3897638 & -0.0196850 & -0.2362205 & 0.1653543 \\ 0.6496063 & -0.3661417 & -0.3937008 & 0.2755906 \\ 0.1929134 & -0.0905512 & -0.0866142 & -0.0393701 \end{pmatrix}\begin{pmatrix} -2 \\ 4 \\ 1.75 \\ -0.5 \end{pmatrix} = \begin{pmatrix} -0.36 \\ -1.35 \\ -3.59 \\ -0.88 \end{pmatrix}$$

$a = -0.36, b = -1.35, c = -3.59, d = -0.88$

4 $F_1 + F_2 + F_3 = 175,000$

$1.025F_1 + 1.048F_2 + 1.035F_3 = 181,615$

and $F_2 = 2F_1$.

Hence form the matrix equation

$$\begin{pmatrix} 1 & 1 & 1 \\ 1.025 & 1.048 & 1.035 \\ 2 & -1 & 0 \end{pmatrix}\begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix} = \begin{pmatrix} 175000 \\ 181615 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1.025 & 1.048 & 1.035 \\ 2 & -1 & 0 \end{pmatrix}^{-1}\begin{pmatrix} 175000 \\ 181615 \\ 0 \end{pmatrix}$$

$F_1 = 30,625$ euros, $F_2 = 61,250$ euros, $F_3 = 83,125$ euros

5 Suppose it is possible to manufacture a, b, c, d of products A, B, C, and D, respectively. Write the problem in terms of matrices:

$$WX = Z, W = \begin{pmatrix} 2 & 1 & 1 & 3 \\ 1 & 3 & 2 & 4 \\ 2 & 1 & 2 & 2 \\ 3 & 4 & 1 & 2 \end{pmatrix}, X = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}, Z = \begin{pmatrix} 240 \\ 380 \\ 280 \\ 400 \end{pmatrix}, X = W^{-1}Z = \begin{pmatrix} 36 \\ 48 \\ 60 \\ 20 \end{pmatrix}.$$

$a = 36, b = 48, c = 60, d = 20$

6 **a** Use the formula for the inverse of 2x2 square matrices to find:

$$(AB)^{-1} = \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix},$$

$$A^{-1}B^{-1} = \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -3 & 4 \end{pmatrix} = \begin{pmatrix} -3 & 5 \\ -1 & 2 \end{pmatrix}'$$

b **i** $B^{-1}A^{-1} = \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix} = (AB)^{-1}$

ii *e.g.* $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$, $B^{-1}A^{-1} = \begin{pmatrix} -2 & 1 \\ \frac{3}{4} & -\frac{1}{4} \end{pmatrix} = (AB)^{-1}$

iii

$$\begin{aligned} (AB)(AB)^{-1} &= I_2 \\ A^{-1}AB(AB)^{-1} &= A^{-1} \\ B(AB)^{-1} &= A^{-1} \\ B^{-1}B(AB)^{-1} &= B^{-1}A^{-1} \\ (AB)^{-1} &= B^{-1}A^{-1}. \end{aligned}$$

7 To decrypt the message, use your GDC to solve for $M = C^{-1}E$, where M is the message, C is the encryption matrix and E is the encrypted message:

$$M = \begin{pmatrix} 61 & 39 & 1 & 20 & 8 & 5 & 13 & 1 & 20 & 9 & 3 & 19 \\ 77 & 9 & 19 & 77 & 20 & 8 & 5 & 77 & 13 & 21 & 19 & 9 \\ 3 & 77 & 15 & 6 & 77 & 18 & 5 & 1 & 19 & 15 & 14 & 61 \end{pmatrix}.$$

Use the conversion table to decrypt the message: "Mathematics is the music of reason" .

8 Follow the procedure given in the textbook.

Exercise 9E

1 **a** Use the reflection matrix with $\alpha = 0$: $T = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

b Use the anticlockwise rotation matrix with $\alpha = \frac{\pi}{4}$: $T = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$.

c Use the anticlockwise rotation matrix with $\alpha = -\frac{\pi}{2}$: $T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

d Use the reflection matrix with $\alpha = \frac{3\pi}{4}$: $T = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$.

e $T = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$.

f $T = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

g $\tan\alpha = \sqrt{3}, \alpha = \frac{\pi}{3}$, so use the reflection matrix to find $T = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$.

2 **a** $A = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$

b $B = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

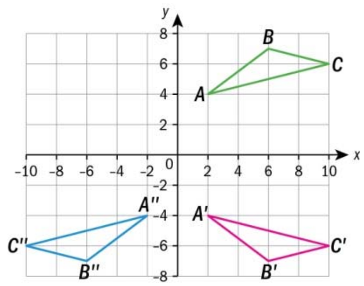
c $AB = \begin{pmatrix} -4 & 0 \\ 0 & -4 \end{pmatrix}$ which is equal to the enlargement matrix with scale factor -4.

3 **a** Reflection across the x-axis matrix is $T = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $TP = \begin{pmatrix} 2 & 6 & 10 \\ -4 & -7 & -6 \end{pmatrix}$

b Reflection across the y-axis matrix is $T' = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$.

The single transformation matrix is $T'T = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$.

Hence, the image of P is $P'' = T'TP = \begin{pmatrix} -2 & -6 & -10 \\ -4 & -7 & -6 \end{pmatrix}$



c Since $TT = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, reflecting the triangle ABC in the x-axis twice will result in the same triangle ABC.

d $T'T = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, which is the same as a rotation by $180°$ (clockwise or anticlockwise). This can also be seen from the graph above.

4 **a** $T = \begin{pmatrix} \cos(2\alpha) & \sin(2\alpha) \\ \sin(2\alpha) & -\cos(2\alpha) \end{pmatrix}$

b $T^2 = \begin{pmatrix} \cos(2\alpha) & \sin(2\alpha) \\ \sin(2\alpha) & -\cos(2\alpha) \end{pmatrix}^2 = \begin{pmatrix} \cos^2(2\alpha) + \sin^2(2\alpha) & 0 \\ 0 & \cos^2(2\alpha) + \sin^2(2\alpha) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

5 **a** $R = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$

b $R^2 = \frac{1}{2}\begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}, R^4 = \frac{1}{4}\begin{pmatrix} -4 & 0 \\ 0 & -4 \end{pmatrix}, R^8 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Since $8 \times 45° = 360°$, R^8 corresponds to rotation by $360°$, i.e. the object stays the same.

6 a The base of the triangle is of length 4 and the height is of length 2, hence the area is $A = \frac{1}{2} \times 4 \times 2 = 4$.

b i $\boldsymbol{T} = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 6 \end{pmatrix}$, $\det \boldsymbol{T} = 18$, $A_1 = 18A = 18 \times 4 = 72$.

ii $\det \boldsymbol{T} = 10 + 6 = 16$, $A_2 = 16A = 64$.

7 a If $\boldsymbol{T} = \begin{pmatrix} \cos(2\alpha) & \sin(2\alpha) \\ \sin(2\alpha) & -\cos(2\alpha) \end{pmatrix}$, $\det \boldsymbol{T} = -\cos^2(2\alpha) - \sin^2(2\alpha) = -1$. New area gets multiplied by $|\det \boldsymbol{T}| = 1$, i.e. doesn't change.

b If $\boldsymbol{T} = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix}$, $\det \boldsymbol{T} = \cos^2(2\alpha) + \sin^2(2\alpha) = 1$. New area gets multiplied by $|\det \boldsymbol{T}| = 1$, i.e. doesn't change.

8 a $\boldsymbol{T}\begin{pmatrix} 14 \\ 12 \end{pmatrix} = \begin{pmatrix} 102 \\ 76 \end{pmatrix}$

b $\boldsymbol{T}^{-1}\begin{pmatrix} -12 \\ 12 \end{pmatrix} = \begin{pmatrix} -54 \\ 30 \end{pmatrix}$

9 a $\boldsymbol{T}\begin{pmatrix} -3 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \\ -11 \end{pmatrix}$, so coordinates are $(2, -11)$

b $\begin{pmatrix} 2 & 2 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -12 \\ 12 \end{pmatrix}$

$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 1 & -2 \end{pmatrix}^{-1}\begin{pmatrix} -12 \\ 12 \end{pmatrix} = \begin{pmatrix} 0 \\ -6 \end{pmatrix}$, so coordinates are $(0, -6)$

10 Let $\boldsymbol{T} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 2 & -3 \end{pmatrix} = \begin{pmatrix} -1 & 4 \\ -1 & 9 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -1 & 4 \\ -1 & 9 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 2 & -3 \end{pmatrix}^{-1}$ Note that on this occasion the matrix needs to be multiplied on the right.

$\Rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 3 & -2 \end{pmatrix}$

11 a $\boldsymbol{R} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\boldsymbol{T} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

b $\boldsymbol{TR} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

c $\boldsymbol{TR}$ is reflection in x axis, so the image is a triangle with vertices $(1, -1), (3, -1), (3, -3)$.

d $\boldsymbol{RT} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, which is a reflection in y axis.

12 a $\boldsymbol{E} = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \end{pmatrix}$ **b** $\boldsymbol{E}'' = \begin{pmatrix} 0.5'' & 0 \\ 0 & 0.5'' \end{pmatrix}$.

13 a $\boldsymbol{Q} = \begin{pmatrix} \cos 40^{\circ} & -\sin 40^{\circ} \\ \sin 40^{\circ} & \cos 40^{\circ} \end{pmatrix}\begin{pmatrix} \cos 20^{\circ} & -\sin 20^{\circ} \\ \sin 20^{\circ} & \cos 20^{\circ} \end{pmatrix}\begin{pmatrix} -2 \\ 5 \end{pmatrix} = \begin{pmatrix} -5.33 \\ 0.77 \end{pmatrix}$.

b $\begin{pmatrix} \cos 60^{\circ} & -\sin 60^{\circ} \\ \sin 60^{\circ} & \cos 60^{\circ} \end{pmatrix} = \begin{pmatrix} \cos 40^{\circ} & -\sin 40^{\circ} \\ \sin 40^{\circ} & \cos 40^{\circ} \end{pmatrix}\begin{pmatrix} \cos 20^{\circ} & -\sin 20^{\circ} \\ \sin 20^{\circ} & \cos 20^{\circ} \end{pmatrix} =$

$\begin{pmatrix} \cos 20^{\circ}\cos 40^{\circ} - \sin 20^{\circ}\sin 40^{\circ} & -\cos 40^{\circ}\sin 20^{\circ} - \sin 40^{\circ}\cos 20^{\circ} \\ \sin 40^{\circ}\cos 20^{\circ} + \cos 40^{\circ}\sin 20^{\circ} & -\sin 40^{\circ}\sin 20^{\circ} + \cos 40^{\circ}\cos 20^{\circ} \end{pmatrix}$.

c Rotation by $\alpha + \theta$ is given by the following matrix:

$$\begin{pmatrix} \cos(\alpha+\theta) & -\sin(\alpha+\theta) \\ \sin(\alpha+\theta) & \cos(\alpha+\theta) \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$
$$= \begin{pmatrix} \cos\theta\cos\alpha - \sin\theta\sin\alpha & -\cos\alpha\sin\theta - \sin\alpha\cos\theta \\ \sin\alpha\cos\theta + \cos\alpha\sin\theta & -\sin\alpha\sin\theta + \cos\alpha\cos\theta \end{pmatrix}$$

Hence $\cos(\alpha+\theta) = \cos\theta\cos\alpha - \sin\theta\sin\alpha, \sin(\alpha+\theta) = \sin\alpha\cos\theta + \cos\alpha\sin\theta$.

Exercise 9F

1 a A′(18,−21), B′(−2,12), C′(6,−18), D′(−14,−29).

b $\begin{pmatrix} -4 & 0 \\ 1 & -4 \end{pmatrix}^{-1}\left(\begin{pmatrix} -6 \\ -3 \end{pmatrix} - \begin{pmatrix} 6 \\ -2 \end{pmatrix}\right) = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$.

2 a $\boldsymbol{X}' = \boldsymbol{TX} = \begin{pmatrix} -2 & 1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} -5 \\ 8 \end{pmatrix} = \begin{pmatrix} 18 \\ 21 \end{pmatrix}$.

b $\boldsymbol{X} = \boldsymbol{T}^{-1}\boldsymbol{X}' = \begin{pmatrix} -\frac{2}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{pmatrix}\begin{pmatrix} 3 \\ 6 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$.

c $\boldsymbol{X}' = \boldsymbol{TX} = \begin{pmatrix} -2 & 1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} a \\ a \end{pmatrix} = \begin{pmatrix} -a \\ a \end{pmatrix}$.

d $\boldsymbol{X} = \boldsymbol{T}^{-1}\boldsymbol{X}' = \begin{pmatrix} -\frac{2}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{pmatrix}\begin{pmatrix} a \\ a \end{pmatrix} = \begin{pmatrix} -\frac{a}{3} \\ \frac{a}{3} \end{pmatrix}$.

3 a $\boldsymbol{T} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 0 & -3 \\ 3 & 0 \end{pmatrix}, \boldsymbol{b} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

b $\begin{pmatrix} x'' \\ y'' \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\left(\begin{pmatrix} 0 & -3 \\ 3 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 2 \\ 3 \end{pmatrix}\right) = \begin{pmatrix} 0 & 3 \\ -3 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -2 \\ -3 \end{pmatrix}$,

i.e. $A = \begin{pmatrix} 0 & 3 \\ -3 & 0 \end{pmatrix}, c = \begin{pmatrix} -2 \\ -3 \end{pmatrix}$.

4 a i $P=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$ **ii** $Q=\begin{pmatrix}\cos135^{\circ}&\sin135^{\circ}\\-\sin135^{\circ}&\cos135^{\circ}\end{pmatrix}$

iii $\tan\alpha=\sqrt{3}$, $\alpha=60^{\circ}$, so use the reflection matrix to find $R=\begin{pmatrix}-\frac{1}{2}&\frac{\sqrt{3}}{2}\\\frac{\sqrt{3}}{2}&\frac{1}{2}\end{pmatrix}$

iv $S=\begin{pmatrix}2&0\\0&4\end{pmatrix}$

b

$$\begin{aligned}A&=SRPQ\\&=\begin{pmatrix}2&0\\0&4\end{pmatrix}\begin{pmatrix}-1/2&\sqrt{3}/2\\\sqrt{3}/2&1/2\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}\cos135^{\circ}&\sin135^{\circ}\\-\sin135^{\circ}&\cos135^{\circ}\end{pmatrix}\\&=\begin{pmatrix}1.9&0.5\\-1.0&3.9\end{pmatrix}.\end{aligned}$$

c $R\left(X+\begin{pmatrix}4\\-2\end{pmatrix}\right)=RX+R\begin{pmatrix}4\\-2\end{pmatrix}=\begin{pmatrix}-0.5&0.866...\\0.866...&0.5\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}+\begin{pmatrix}-3.73\\2.46\end{pmatrix}.$

5 a The side of T_0 is 2 and it multiplies by 0.5 with each transformation. Hence, the total perimeter $2+2+1+1+0.5+0.5+0.25+0.25+0.25=7.75$.

b Maximum perimeter is reached when six triangles each having angles of 60° add up to the full revolution 360°. i.e. T_6 is inside of T_0. Then, the total perimeter is $(2-0.0625)+2+1+1+0.5+0.5+0.25+0.25+0.125+0.125+0.0625+0.0625=7.8125$.

c i $E=\begin{pmatrix}0.5&0\\0&0.5\end{pmatrix}$, $\det E=0.25$. **ii** $T_0=\frac{1}{2}\times2\times2\times\sin60^{\circ}=\sqrt{3}$

iii This is a geometric series with $u_1=T_0=\sqrt{3}$ and $r=\det(E)=0.25$. Hence,

$$A=S_5=\frac{\sqrt{3}\left(1-0.25^{6}\right)}{(1-0.25)}=2.31$$

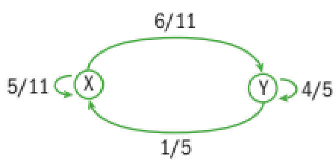
d $R=\begin{pmatrix}\frac{1}{2}&\frac{\sqrt{3}}{2}\\-\frac{\sqrt{3}}{2}&\frac{1}{2}\end{pmatrix}$ or $\begin{pmatrix}0.5&0.866\\-0.866&0.5\end{pmatrix}$

e $ER=\begin{pmatrix}\frac{1}{4}&\frac{\sqrt{3}}{4}\\-\frac{\sqrt{3}}{4}&\frac{1}{4}\end{pmatrix}$ or $\begin{pmatrix}0.25&0.433\\-0.433&0.25\end{pmatrix}$

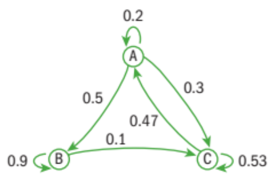
f $(ER)^8\begin{pmatrix}-2&-1&0\\0&\sqrt{3}&0\end{pmatrix}=\begin{pmatrix}0.00391&0.00781&0\\0.00677&0&0\end{pmatrix}.$

Exercise 9G

1 a



b



2 a $T = \begin{pmatrix} 0.75 & 0.2 \\ 0.25 & 0.8 \end{pmatrix}$

b $T^3 = \begin{pmatrix} 0.54 & 0.37 \\ 0.46 & 0.63 \end{pmatrix}$, so the probability that a person who buys Popsi now will change to Ceko three weeks from now is 0.37.

3 a $T = \begin{pmatrix} 0.65 & 0.26 & 0.05 \\ 0.25 & 0.70 & 0.05 \\ 0.10 & 0.04 & 0.90 \end{pmatrix}$

b $T^5 = \begin{pmatrix} 0.36 & 0.37 & 0.17 \\ 0.37 & 0.41 & 0.18 \\ 0.28 & 0.22 & 0.64 \end{pmatrix}$, so the probability that a person buying neither Ceko nor Popsi now will buy Popsi 5 weeks from now is 0.18.

4 a $T = \begin{pmatrix} 0 & \frac{1}{4} & 0 \\ 1 & \frac{1}{2} & 1 \\ 0 & \frac{1}{4} & 0 \end{pmatrix}$

b There are 4 states:

1. 3W-3B
2. 2W1B-1W2B
3. 1W2B-2W1B
4. 3B-3W

and the transition matrix is $T = \begin{pmatrix} 0 & \frac{1}{9} & 0 & 0 \\ 1 & \frac{4}{9} & \frac{4}{9} & 0 \\ 0 & \frac{4}{9} & \frac{4}{9} & 1 \\ 0 & 0 & \frac{1}{9} & 0 \end{pmatrix}$.

c Consider the general case n W- m B with $m \le n$ without loss of generality. Then, the transition matrix is of size $(m+1)\times(m+1)$. States can transition to the previous state, the current state or the next state, if these exist. Thus, element $T_{2,1} = 1$ and all the other entries in the first column are 0. Element $T_{m,m+1} = 1$ and all the other entries in the last column are 0. In an inner column, only elements $T_{j-1,j}$, $T_{j,j}$ and $T_{j+1,j}$ are non-zero.

Consider $j-1$ blue coins in the box with n coins and $j-1$ white coins in the box with m coins. Then, the probability of moving to the previous state with $j-2$ blue coins in the box with n coins is $T_{j-1,j} = \frac{(j-1)^2}{nm}$. Similarly, the probability of moving to the next state with j blue coins in the box with n coins is $T_{j+1,j} = \frac{(n-j+1)(m-j+1)}{nm}$. Finally, staying in the same state has probability $T_{j,j} = 1 - \frac{(j-1)^2}{nm} - \frac{(n-j+1)(m-j+1)}{nm}$.

5 a $X = T^n B$, $T = \begin{pmatrix} 0.959 & 0.032 \\ 0.041 & 0.968 \end{pmatrix}$, $B = \begin{pmatrix} 45520 \\ 38745 \end{pmatrix}$, $X = \begin{pmatrix} a \\ b \end{pmatrix}$, where a, b is the number of people planning to vote for candidates A and B, respectively, after n weeks.

b $X = T^5 B = \begin{pmatrix} 42813 \\ 41452 \end{pmatrix}$, so candidate A would win by 1361 votes which is 1.62% of all votes

Exercise 9H

1 a $X = \begin{pmatrix} 0.45 & 0.45 \\ 0.55 & 0.55 \end{pmatrix}$

b $X = \begin{pmatrix} 0.14 & 0.14 & 0.14 \\ 0.51 & 0.51 & 0.51 \\ 0.35 & 0.35 & 0.35 \end{pmatrix}$.

c $X = \begin{pmatrix} 0.35 & 0.35 & 0.35 & 0.35 \\ 0.10 & 0.10 & 0.10 & 0.10 \\ 0.25 & 0.25 & 0.25 & 0.25 \\ 0.29 & 0.29 & 0.29 & 0.29 \end{pmatrix}$.

2 a

$$\begin{aligned} 0.81u_1 + 0.1u_2 + 0.08u_3 &= u_1 \\ 0.09u_1 + 0.15u_2 + 0.5u_3 &= u_2 \\ 0.1u_1 + 0.75u_2 + 0.42u_3 &= u_3 \\ u_1 + u_2 + u_3 &= 1 \end{aligned}$$

Solving

$$u = \begin{pmatrix} 0.32 \\ 0.27 \\ 0.41 \end{pmatrix}$$

b The long-term probability matrix is $X = \begin{pmatrix} 0.32 & 0.32 & 0.32 \\ 0.27 & 0.27 & 0.27 \\ 0.41 & 0.41 & 0.41 \end{pmatrix}$. Then, $Xu = u$.

3 a $\begin{pmatrix} A \\ B \\ C \end{pmatrix} = T^2 \begin{pmatrix} 45 \\ 25 \\ 30 \end{pmatrix} = \begin{pmatrix} 34.6 \\ 36.5 \\ 28.9 \end{pmatrix}\%$.

b Long term matrix $X = \begin{pmatrix} 0.2085 & 0.2085 & 0.2085 \\ 0.5787 & 0.5787 & 0.5787 \\ 0.2128 & 0.2128 & 0.2128 \end{pmatrix}$ and $\begin{pmatrix} A \\ B \\ C \end{pmatrix} = X \begin{pmatrix} 45 \\ 25 \\ 30 \end{pmatrix} = \begin{pmatrix} 20.85 \\ 57.87 \\ 21.28 \end{pmatrix}\%$, assuming that the transition probabilities remain constant in time. Therefore, the long-term percentage of customers who will be provided electricity by B is 57.9%.

c $T^2 = \begin{pmatrix} 0.6508 & 0.0878 & 0.1034 \\ 0.2144 & 0.8554 & 0.1832 \\ 0.1348 & 0.0568 & 0.7134 \end{pmatrix}$, $T^5 = \begin{pmatrix} 0.3942 & 0.1546 & 0.1732 \\ 0.3942 & 0.7267 & 0.3570 \\ 0.2116 & 0.1187 & 0.4698 \end{pmatrix}$,

$T^{20} = \begin{pmatrix} 0.2113 & 0.2072 & 0.2092 \\ 0.5714 & 0.5856 & 0.5673 \\ 0.2173 & 0.2072 & 0.2234 \end{pmatrix}$.

We see that the convergence to the long term matrix is slow even when the transition probabilities remain constant. Realistically, once the electricity providers notice the trend, they will respond to these changes quickly by changing the price or quality of their services to prevent customers from switching in which case the transition probabilities are unlikely to stay constant.

4 a $F_{1,2} = F_{1,3} = 0$: No cars that need a minor or major repair will be functioning normally after 1 month.

$F_{2,3} = F_{2,4} = 0$: No cars that need a major repair or are broken down will need a minor repair after 1 month.

$F_{3,4} = 0$: No cars that are broken down will need a major repair after 1 month.

$F_{4,4} = 0.04$: These cars are broken down and remain broken down after a month. They require more than a month to fix.

b Check that all columns sum to 1.

c $\begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} = F^{10} \begin{pmatrix} 500 \\ 25 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 388 \\ 70 \\ 28 \\ 39 \end{pmatrix}$.

d The company should repair functioning cars which need a minor or major repair before they are fully broken down to reduce the number of broken down cars.

Exercise 9I

1 a $\lambda^2 + 3\lambda + 2 = 0$, $\lambda_1 = -2$, $\lambda_2 = -1$. **b** $\lambda^2 - 25 = 0$, $\lambda_1 = 5$, $\lambda_2 = -5$.

c $\lambda^2 - 3 = 0,\ \lambda_1 = \sqrt{3},\ \lambda_2 = -\sqrt{3}.$

2 Characteristic equation of the matrix is $\lambda^2 - 11\lambda + 18 = 0$, but $\lambda = -1$ does not satisfy it. Hence, it is not an eigenvalue of the matrix.

3 **a** Find the eigenvalues from $\lambda^2 - 2\lambda + 1 = 0,\ \lambda_{1,2} = 1$. Then, $\boldsymbol{C}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$ gives $y = \frac{1}{2}x$, so possible eigenvector is $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

b Find the eigenvalues from $\lambda^2 - 5\lambda - 14 = 0,\ \lambda_1 = -2, \lambda_2 = 7$. Then, $\boldsymbol{Q}\begin{pmatrix} x \\ y \end{pmatrix} = -2\begin{pmatrix} x \\ y \end{pmatrix}$ gives $y = -x$ and $\boldsymbol{Q}\begin{pmatrix} x \\ y \end{pmatrix} = 7\begin{pmatrix} x \\ y \end{pmatrix}$ gives $y = \frac{5}{4}x$, so possible eigenvectors are $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ 5 \end{pmatrix}$.

c Find the eigenvalues from $\lambda^2 - 2\lambda + 1 = 0,\ \lambda_{1,2} = 1$. Then, $\boldsymbol{I}_2\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$, so possible eigenvectors are $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

d Find the eigenvalues from $\lambda^2 - 9 = 0,\ \lambda_1 = -3, \lambda_2 = 3$. Then, $\boldsymbol{P}\begin{pmatrix} x \\ y \end{pmatrix} = -3\begin{pmatrix} x \\ y \end{pmatrix}$ gives $y = x$ and $\boldsymbol{P}\begin{pmatrix} x \\ y \end{pmatrix} = 3\begin{pmatrix} x \\ y \end{pmatrix}$ gives $y = -x$, so possible eigenvectors are $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

e Find the eigenvalues from $\lambda^2 - 0.7\lambda - 0.3 = 0,\ \lambda_1 = 1, \lambda_2 = -0.3$. Then, $\boldsymbol{T}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$ gives $y = \frac{6}{7}x$ and $\boldsymbol{T}\begin{pmatrix} x \\ y \end{pmatrix} = -0.3\begin{pmatrix} x \\ y \end{pmatrix}$ gives $y = -x$, so possible eigenvectors are $\begin{pmatrix} 7 \\ 6 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

4 **a** The eigenvalue equation is $\lambda^2 + (\boldsymbol{b} - \boldsymbol{a} - 1)\lambda + (\boldsymbol{a} - \boldsymbol{b}) = 0$. If $\lambda = 1$, we get

$\text{LHS} = 1 + (b - a - 1) + (a - b) = 0 = \text{RHS}$. Hence, $\lambda = 1$ is a solution.

The equation can be factorised as $(\lambda - 1)(\lambda - (a - b)) = 0$ hence other eigen value is $a - b$

b Then, $\boldsymbol{T}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$ gives $y = \frac{1-a}{b}x$ and $\boldsymbol{T}\begin{pmatrix} x \\ y \end{pmatrix} = (a - b)\begin{pmatrix} x \\ y \end{pmatrix}$ gives $y = -x$, so possible eigenvectors are $\begin{pmatrix} b \\ 1-a \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

Exercise 9J

1 Eigenvectors are multiples of $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ for $\lambda = -1$ and multiples of $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ for $\lambda = -4$.

Hence, $\boldsymbol{R} = \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & -4 \end{pmatrix}\begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{pmatrix}$.

2 **a** Find the eigenvalues from $\lambda^2 - 3\lambda + 2 = 0$, $\lambda_1 = 1, \lambda_2 = 2$. Then, $\boldsymbol{A}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$ gives $y = \frac{3}{7}x$ and $\boldsymbol{A}\begin{pmatrix} x \\ y \end{pmatrix} = 2\begin{pmatrix} x \\ y \end{pmatrix}$ gives $y = \frac{2}{5}x$, so possible eigenvectors are $\begin{pmatrix} 7 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 5 \\ 2 \end{pmatrix}$. Matrix $\boldsymbol{A}$ is diagonalisable because it has two distinct eigenvalues.

b $\boldsymbol{A} = \begin{pmatrix} 7 & 5 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} -2 & 5 \\ 3 & -7 \end{pmatrix}$.

c $A^n = \begin{pmatrix} 7 & 5 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 2^n \end{pmatrix}\begin{pmatrix} -2 & 5 \\ 3 & -7 \end{pmatrix} = \begin{pmatrix} 15\times 2^n - 14 & 35\left(1-2^n\right) \\ 6\left(2^n - 1\right) & 15 - 14\times 2^n \end{pmatrix}$.

d $\boldsymbol{A}^4 = \begin{pmatrix} 226 & -525 \\ 90 & -209 \end{pmatrix}$.

3 **a** Find the eigenvalues from $\lambda^2 - 0.65\lambda - 0.35 = 0$, $\lambda_1 = 1$, $\lambda_2 = -0.35$. Then, $\boldsymbol{T}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$ gives $y = \frac{4}{5}x$ and $\boldsymbol{T}\begin{pmatrix} x \\ y \end{pmatrix} = -0.35\begin{pmatrix} x \\ y \end{pmatrix}$ gives $y = -x$, so possible eigenvectors are $\begin{pmatrix} 5 \\ 4 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

Hence, $\boldsymbol{T} = \begin{pmatrix} 5 & 1 \\ 4 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -0.35 \end{pmatrix}\begin{pmatrix} \frac{1}{9} & \frac{1}{9} \\ \frac{4}{9} & -\frac{5}{9} \end{pmatrix}$.

b $\boldsymbol{T}^4 = \begin{pmatrix} 5 & 1 \\ 4 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & (-0.35)^4 \end{pmatrix}\begin{pmatrix} \frac{1}{9} & \frac{1}{9} \\ \frac{4}{9} & -\frac{5}{9} \end{pmatrix} = \begin{pmatrix} 0.562 & 0.547 \\ 0.438 & 0.453 \end{pmatrix}$.

c $\boldsymbol{T}^n \to \begin{pmatrix} 0.56 & 0.56 \\ 0.44 & 0.44 \end{pmatrix}$ as $n \to \infty$.

4 **a** $\boldsymbol{T} = \begin{pmatrix} \frac{1}{4} & \frac{1}{3} \\ \frac{3}{4} & \frac{2}{3} \end{pmatrix}$.

b $\boldsymbol{D} = \boldsymbol{T}^2\begin{pmatrix} 6500 \\ 5200 \end{pmatrix} = \begin{pmatrix} 3620 \\ 8080 \end{pmatrix}$ – more customers choose company S.

c Find the eigenvalues from $\lambda^2 - \frac{11}{12}\lambda - \frac{1}{12} = 0$, $\lambda_1 = 1, \lambda_2 = -\frac{1}{12}$. Then, $\boldsymbol{T}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$ gives $y = \frac{9}{4}x$ and $\boldsymbol{T}\begin{pmatrix} x \\ y \end{pmatrix} = -\frac{1}{12}\begin{pmatrix} x \\ y \end{pmatrix}$ gives $y = -x$, so possible eigenvectors are $\begin{pmatrix} 4 \\ 9 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

Hence, $\boldsymbol{T} = \begin{pmatrix} 4 & 1 \\ 9 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -\frac{1}{12} \end{pmatrix}\begin{pmatrix} \frac{1}{13} & \frac{1}{13} \\ \frac{9}{13} & -\frac{4}{13} \end{pmatrix}$.

d $T^n = \begin{pmatrix} 4 & 1 \\ 9 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \left(-\frac{1}{12}\right)^n \end{pmatrix}\begin{pmatrix} \frac{1}{13} & \frac{1}{13} \\ \frac{9}{13} & -\frac{4}{13} \end{pmatrix} = \frac{1}{13}\begin{pmatrix} 4+9p^n & 4-4p^n \\ 9-9p^n & 9+4p^n \end{pmatrix}$, $p = -\frac{1}{12}$.

e $T^n \begin{pmatrix} 6500 \\ 5200 \end{pmatrix} = \begin{pmatrix} 2900p^n + 3600 \\ 8100 - 2900p^n \end{pmatrix}$

Hence, $R_n = 2900p^n + 3600$

f For $n = 2$, $R_2 = \dfrac{2900}{(-12)^2} + 3600 = 3620$, which agrees with the answer in **b**.

g As $n \to \infty$, $R_n \to 3600$. So the long term number of customers buying from R is 3600.

Chapter review

1 a $2C + 3D = \begin{pmatrix} -11 & 9 \\ -4 & -7 \end{pmatrix}$

b Not possible because the dimensions of the matrices don't agree.

c $DF = \begin{pmatrix} -4 & -2 & 7 \\ 5 & -20 & 10 \end{pmatrix}$

d $FE = \begin{pmatrix} 4 & 7 & -11 \\ -1 & 4 & -9 \end{pmatrix}$

e $E^2 = \begin{pmatrix} -6 & -1 & 3 \\ 0 & -3 & -2 \\ -7 & -11 & 13 \end{pmatrix}$

f $(C + D)^2 = \begin{pmatrix} 8 & -20 \\ 10 & -7 \end{pmatrix}$

g $EB + B = \begin{pmatrix} 0 \\ -4 \\ 8 \end{pmatrix}$

h $C^{-1} = \begin{pmatrix} 2 & -1.5 \\ 1 & -0.5 \end{pmatrix}$

i $CDA = \begin{pmatrix} 47 \\ 74 \end{pmatrix}$

j $C - 2I = \begin{pmatrix} -3 & 3 \\ -2 & 2 \end{pmatrix}$

2

$3a - 4 = 8, \quad a = 4.$

$2 - 8b = 6, \quad b = -\frac{1}{2}.$

It is assumed that the rest of the equation is consistent.

3 **a** $X = \begin{pmatrix} \frac{5}{3} & \frac{4}{3} \\ -2 & -\frac{2}{3} \end{pmatrix}$

b $X = \begin{pmatrix} 0.4 & -0.6 \\ 0.2 & 0.2 \end{pmatrix}\begin{pmatrix} -5 & 2 \\ 1 & -10 \end{pmatrix} = \begin{pmatrix} -2.6 & 6.8 \\ -0.8 & -1.6 \end{pmatrix}$

c $X = \begin{pmatrix} 2 & 4 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 1 & 0.5 \\ 2 & 1.5 \end{pmatrix} = \begin{pmatrix} 10 & 7 \\ 8 & 5.5 \end{pmatrix}.$

4 **a** $\boldsymbol{AX} = \boldsymbol{B}, \boldsymbol{A} = \begin{pmatrix} -4 & 7 \\ 3 & -2 \end{pmatrix}, \boldsymbol{X} = \begin{pmatrix} x \\ y \end{pmatrix}, \boldsymbol{B} = \begin{pmatrix} -20 \\ 10 \end{pmatrix}, \boldsymbol{X} = \boldsymbol{A}^{-1}\boldsymbol{B} = \begin{pmatrix} \frac{30}{13} \\ -\frac{20}{13} \end{pmatrix}.$

b $\boldsymbol{AX} = \boldsymbol{B}, \boldsymbol{A} = \begin{pmatrix} 1 & -2 & -2 \\ 1 & 3 & -4 \\ 4 & 5 & -2 \end{pmatrix}, \boldsymbol{X} = \begin{pmatrix} w \\ y \\ z \end{pmatrix}, \boldsymbol{B} = \begin{pmatrix} 3 \\ 6 \\ 3 \end{pmatrix}, \boldsymbol{X} = \boldsymbol{A}^{-1}\boldsymbol{B} = \begin{pmatrix} 0 \\ 0 \\ -1.5 \end{pmatrix}.$

c $AX = \boldsymbol{B}, \boldsymbol{A} = \begin{pmatrix} 6 & -5 \\ -9 & 7.5 \end{pmatrix}, \boldsymbol{X} = \begin{pmatrix} x \\ y \end{pmatrix}, \boldsymbol{B} = \begin{pmatrix} 18 \\ -27 \end{pmatrix}$

However, $|A| = 0$, so A has no inverse. The two equations represent the same line. There is no solution.

5 **a** $\boldsymbol{X}' = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}\left(\boldsymbol{X} + \begin{pmatrix} -3 \\ 5 \end{pmatrix}\right) = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}\boldsymbol{X} + \begin{pmatrix} -\frac{3}{2} \\ -\frac{5}{2} \end{pmatrix}, \boldsymbol{A} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, \boldsymbol{b} = \begin{pmatrix} -\frac{3}{2} \\ -\frac{5}{2} \end{pmatrix}.$

b Apply the above transformation to $\boldsymbol{X} = \begin{pmatrix} -4 & -1 & 8 & 5 \\ -2 & 7 & 4 & -5 \end{pmatrix}$ to get P′(−3.5,−1.5), Q′(−2,−6), R′(2.5,−4.5) and S′(1,0).

c $k = |\det A| = \left|-\frac{1}{4}\right| = 0.25.$

6 **a** $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 13 \\ 25 \end{pmatrix}$

b $\begin{pmatrix} x \\ y \end{pmatrix} = -\frac{1}{11}\begin{pmatrix} 4 & -1 \\ 1 & -3 \end{pmatrix}\left[\begin{pmatrix} -7 \\ 0 \end{pmatrix} - \begin{pmatrix} -1 \\ 2 \end{pmatrix}\right] = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$

c

$$\begin{aligned}
\begin{pmatrix} x' \\ y' \end{pmatrix} &= \begin{pmatrix} -3 & 1 \\ -1 & 4 \end{pmatrix}\begin{pmatrix} a \\ 2a \end{pmatrix} + \begin{pmatrix} -1 \\ 2 \end{pmatrix} \\
&= \begin{pmatrix} -a \\ 7a \end{pmatrix} + \begin{pmatrix} -1 \\ 2 \end{pmatrix} \\
&= \begin{pmatrix} -a-1 \\ 7a+2 \end{pmatrix}
\end{aligned}$$

7 a i $k = 0.5$ since the longest side of the triangle is reduced from 8 to 2 in two enlargements.

And from the diagram, $\theta = 45^o$. Hence, as $\sin 45^\circ = \cos 45^\circ = \dfrac{1}{\sqrt{2}}$

$$A = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \end{pmatrix} \times \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} \\ \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} \\ \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} \end{pmatrix} \text{ or } \begin{pmatrix} 0.354 & -0.354 \\ 0.354 & 0.354 \end{pmatrix}$$

ii $$T_3 = A^3 \begin{pmatrix} 0 & 4 & 8 \\ 0 & 4 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}. \text{ or } \begin{pmatrix} 0 & -0.707 & -0.707 \\ 0 & 0 & 0.707 \end{pmatrix}$$

b i $$C_1 = A = \frac{\sqrt{2}}{4} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}. \text{ or } \begin{pmatrix} 0.354 & -0.354 \\ 0.354 & 0.354 \end{pmatrix}$$

ii

$$C_2 = A^2 = \frac{2}{16} \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\frac{1}{4} \\ \frac{1}{4} & 0 \end{pmatrix}$$

$$C_3 = A^3 = \frac{2\sqrt{2}}{64} \begin{pmatrix} -2 & -2 \\ 2 & -2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{8\sqrt{2}} & -\frac{1}{8\sqrt{2}} \\ \frac{1}{8\sqrt{2}} & -\frac{1}{8\sqrt{2}} \end{pmatrix} = \begin{pmatrix} -0.0884 & -0.0884 \\ 0.0884 & -0.0884 \end{pmatrix}$$

$$C_4 = A^4 = \frac{4}{256} \begin{pmatrix} -4 & 0 \\ 0 & -4 \end{pmatrix} = \begin{pmatrix} -\frac{1}{16} & 0 \\ 0 & -\frac{1}{16} \end{pmatrix} = \begin{pmatrix} -0.0625 & 0 \\ 0 & -0.0625 \end{pmatrix}$$

iii The maximal area is obtained for triangles up to T_7 :

$$A = A_0 \left(1 + |\det C_1| + |\det C_2| + \ldots + |\det C_7|\right)$$
$$= \frac{1}{2}(4 \times 8)\left(1 + \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \frac{1}{256} + \frac{1}{1024} + \frac{1}{4096} + \frac{1}{16384}\right)$$
$$= 16 \times 1 \times \frac{1-(1/4)^8}{1-1/4}$$
$$= 21.3$$

Exam-style questions

8 a $2B = \begin{pmatrix} 8 & 0 \\ 2 & -4 \end{pmatrix}$ **b** $A - B = \begin{pmatrix} -1 & -2 \\ 1 & -1 \end{pmatrix}$ c $AB = \begin{pmatrix} 10 & 4 \\ 5 & 6 \end{pmatrix}$ **d** $(A+B)^2 = \begin{pmatrix} 43 & -4 \\ 6 & 19 \end{pmatrix}$.

9 Find the determinant:

$$x(3-x) - (-1) \times 4 = 0$$
$$-x^2 + 3x + 4 = 0$$
$$x = -1 \text{ or } x = 4$$

10 a P is a rotation of 60° while Q is a rotation of −30°. Thus, PQ is a rotation of 30°. Since 3 × 30 = 90°,

$$(\boldsymbol{PQ})^3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

b

$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ is for a rotation of 180°. Hence, n = 6.

11 a $\boldsymbol{A}^2 - 10\boldsymbol{A} + 21\boldsymbol{I} = \begin{pmatrix} 19 & 10 \\ 30 & 39 \end{pmatrix} - \begin{pmatrix} 40 & 10 \\ 30 & 60 \end{pmatrix} + \begin{pmatrix} 21 & 0 \\ 0 & 21 \end{pmatrix} = 0.$

b Since $\boldsymbol{A}^2 = 10\boldsymbol{A} - 21\boldsymbol{I}$,

$\boldsymbol{A}^3 = 10\boldsymbol{A}^2 - 21\boldsymbol{A} = (100\boldsymbol{A} - 210I) - 21\boldsymbol{A} = 79\boldsymbol{A} - 210\boldsymbol{I}.$

c $\boldsymbol{A}^4 = 79\boldsymbol{A}^2 - 210\boldsymbol{A} = 790\boldsymbol{A} - 1659\boldsymbol{I} - 210\boldsymbol{A} = 580\boldsymbol{A} - 1659\boldsymbol{I}.$

12 a $\boldsymbol{A}^{-1} = \begin{pmatrix} 0.125 & -0.875 & 0.625 \\ 0.25 & 1.25 & -0.75 \\ -0.125 & -0.125 & 0.375 \end{pmatrix}$

b $\boldsymbol{C} = \boldsymbol{A}^{-1}\boldsymbol{B} = \begin{pmatrix} 0.75 & -0.875 & 0.75 \\ -0.5 & 1.25 & -0.5 \\ 0.25 & -0.125 & 0.25 \end{pmatrix}.$

13 a $\boldsymbol{A} = \begin{pmatrix} 5 & 0 & 3 \\ 1 & -2 & 5 \\ 0 & 3 & 7 \end{pmatrix}, \boldsymbol{X} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \boldsymbol{B} = \begin{pmatrix} 23 \\ 23 \\ 122 \end{pmatrix}.$

b $\boldsymbol{A}^{-1} = \begin{pmatrix} \frac{29}{136} & -\frac{9}{136} & -\frac{3}{68} \\ \frac{7}{136} & -\frac{25}{136} & \frac{11}{68} \\ -\frac{3}{136} & \frac{15}{136} & \frac{5}{68} \end{pmatrix}$

c $\boldsymbol{X} = \boldsymbol{A}^{-1}\boldsymbol{B} = \begin{pmatrix} -2 \\ 15 \\ 11 \end{pmatrix}.$

14 Suppose, x adults, y children and z OAPs attended the screening. Then, we can set up the following system, where the first row is the number of people, the second row is the price for a person and the third row represents the number of OAPs attending: $\boldsymbol{AX} = \boldsymbol{B}$ with $\boldsymbol{A} = \begin{pmatrix} 1 & 1 & 1 \\ 8 & 5 & 4 \\ 0 & 0 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 750 \\ 4860 \\ 750 \end{pmatrix}$. Solution $\boldsymbol{X} = \boldsymbol{A}^{-1}\boldsymbol{B} = \begin{pmatrix} 420 \\ 180 \\ 150 \end{pmatrix}$, i.e. 180 children attended the screening.

15 a Find the eigenvalues from $\lambda^2 - 8\lambda + 12 = 0$, $\lambda_1 = 2, \lambda_2 = 6$. Then, $\boldsymbol{A}\begin{pmatrix} x \\ y \end{pmatrix} = 2\begin{pmatrix} x \\ y \end{pmatrix}$ gives $y = x$ and $\boldsymbol{A}\begin{pmatrix} x \\ y \end{pmatrix} = 6\begin{pmatrix} x \\ y \end{pmatrix}$ gives $y = -x$, so possible eigenvectors are $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

b $\boldsymbol{D} = \begin{pmatrix} 2 & 0 \\ 0 & 6 \end{pmatrix}, \boldsymbol{P} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$

c $\boldsymbol{A}^n = \boldsymbol{PD}^n\boldsymbol{P}^{-1}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 2^n & 0 \\ 0 & 6^n \end{pmatrix}\begin{pmatrix} 0.5 & 0.5 \\ 0.5 & -0.5 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2^n + 6^n & 2^n - 6^n \\ 2^n - 6^n & 2^n + 6^n \end{pmatrix}.$

10 Analyzing rates of change: differential calculus

Skills check

1 Gradient of $y = 2x - 3$ is 2. Line perpendicular to y has gradient $-\frac{1}{2}$. Hence, the line going through (3,4) will be $y = -\frac{1}{2}x + \frac{11}{2}$.

2 **a** $x^{2-7} = x^{-5}$, b x^{-1}, c $3x^{-\frac{1}{2}}$

3 Plot the curve using GDC. It is decreasing on $0 \le x \le 2$, so max value y = 20 occurs at x = 0, and min value y = -20 occurs at x = 2.

Exercise 10A

1 $f'(x) = \lim_{h\to 0}\left(\frac{4(x+h)-4x}{h}\right) = 4$

2 $f'(x) = \lim_{h\to 0}\left(\frac{-2-(-2)}{h}\right) = 0$

3 $f'(x) = \lim_{h\to 0}\left(\frac{3(x+h)-5-3x+5}{h}\right) = 3$

4 $f'(x) = \lim_{h\to 0}\left(\frac{c-c}{h}\right) = 0$

5 $f'(x) = \lim_{h\to 0}\left(\frac{m(x+h)-mx}{h}\right) = m$

6 $f'(x) = \lim_{h\to 0}\left(\frac{x^2+2xh+h^2-3x-3h+7-(x^2-3x+7)}{h}\right) = \lim_{h\to 0}\left(\frac{2xh+h^2-3h}{h}\right)$

$= \lim_{h\to 0}(2x+h-3) = 2x-3$

7 $f'(x) = \lim_{h\to 0}\left(\frac{2x^2+4xh+2h^2-2x^2}{h}\right) = \lim_{h\to 0}(4x+2h) = 4x$

8 $f'(x) = \lim_{h\to 0}\left(\frac{x^4+4x^3h+6x^2h^2+4xh^3+h^4-x^4}{h}\right) = \lim_{h\to 0}\left(4x^3+6x^2h+4xh^2+h^3\right) = 4x^3$

9 $f'(x) = \lim_{h\to 0}\left(\frac{\frac{1}{x+h}-\frac{1}{x}}{h}\right) = \lim_{h\to 0}\left(\frac{-h}{xh(x+h)}\right) = -\frac{1}{x^2}$

10 $f'(x) = anx^{n-1}$

Exercise 10B

1 **a** $\frac{dy}{dx} = 0$, when x = 2, $\frac{dy}{dx} = 0$

b $\frac{dy}{dx} = 4$, when x = 2, $\frac{dy}{dx} = 4$

c $f'(x) = 6x, f'(2) = 12$

d $f'(x) = 10x - 3, f'(2) = 17$

e $f'(x) = 12x^3 + 7, f'(2) = 12 \times 8 + 7 = 103$

f $f'(x) = 20x^3 - 6x + 2, f'(2) = 20 \times 8 - 12 + 2 = 150.$

2 **a** $f'(x) = \frac{1}{3}\sqrt[3]{x^{-2}}$, $f'(1) = \frac{1}{3}$

b $\frac{dy}{dx} = \frac{1}{\sqrt{x}}$, when x = 1, $\frac{dy}{dx} = 1$

c $\frac{dy}{dx} = 4x + \frac{3}{x^2}$, when x = 1, $\frac{dy}{dx} = 7$

d $\frac{dy}{dx} = -\frac{18}{x^4} + 4$ when x = 1, $\frac{dy}{dx} = -14$

e $\frac{dy}{dx} = -\frac{21}{x^4} + 32x^3 - 12x$, when x = 1, $\frac{dy}{dx} = -1.$

3 **a** $\frac{ds}{dt} = 4 + \frac{4}{t^{\frac{3}{2}}}$, when t = 16, $\frac{ds}{dt} = 4\frac{1}{16}$

b $\frac{dv}{dt} = 3t^{-\frac{1}{4}} + 4t^{-\frac{5}{4}}$, when t = 16, $\frac{dv}{dt} = \frac{3}{2} + \frac{4}{32} = \frac{52}{32}$.

4 **a** **i** $y = 6x^2 + 5x - 4, \frac{dy}{dx} = 12x + 5$.

ii $\frac{dy}{dx} > 0$, i.e. function is increasing for $12x > -5, x > -\frac{5}{12}$.

b **i** $f(x) = 2x^4 + 8x^2 - 10x, f'(x) = 8x^3 + 16x - 10$

ii $f'(x) > 0$, i.e. function is increasing for $x > 0.544$ (use GDC to find the where f′(x) crosses zero).

c **i** $g'(x) = 3x^2 + 6x - 9$, $g'(x) = 0$ when $x = 1, -3$

ii $g'(x) > 0$ for $x < -3$ and $x > 1$.

5 **a** $\frac{dA}{dr} = 2\pi r$

b when r = 2, $\frac{dA}{dr} = 4\pi$

6 **a** $\frac{dP}{dc} = -0.112c + 5.6$

b when c = 20, $\frac{dP}{dc} = 3.36$, when c = 60, $\frac{dP}{dc} = -1.12$.

c When the derivative is positive, increasing the number of cupcakes sold would increase the profit, but when the derivative is negative, increasing the number of cupcakes sold would decrease the profit.

7 **a** $f'(t) = 10 - 10t$

b Rate of change of distance, i.e. speed.

c $f'(0.5) = 10 - 5 = 5, f'(1.5) = 10 - 15 = -5$. A positive value represents the bungee jumper going downwards while a negative value represents the bungee jumper going upwards.

d $f(2) = 0$, so the model suggests that the bungee jumper returns to his starting point, which in practice does not occur.

8 $f'(x) = 3x^2 + 2x + 2 = 3$ at A and B. Hence, solve $3x^2 + 2x - 1 = 0$, i.e. $x = -1$ or $\frac{1}{3}$.

For $x = -1$, $y = -2$. For $x = \frac{1}{3}$, $y = 0.815$. Hence, A and B are (-1,-2) and (1/3, 0.815).

9 Find a point x at which the gradient of h is equal to $\tan(45°) = 1$. $h'(x) = 2 - 0.2x = 1$ when x = 5. Finally, $h(5) = 10 - 2.5 = 7.5$.

Exercise 10C

1 $f'(x) = 4x, f'(3) = 12$. At the point, $f(3) = 14$, so the equation of the tangent is $y - 14 = 12(x - 3)$ or $y = 12x - 22$.

2 $f'(x) = -2x + 2, f'(1) = 0$. At the point, $f(1) = 1$, so the equation of the tangent is $y - 1 = 0$ or $y = 1$.

3 Gradient at the point: $f'(x) = 6x - 4$, $f'(1) = 2$. Gradient of the normal to the graph is $-\frac{1}{2}$.

Equation of the normal goes through the point (1,4) so it is $y = -\frac{1}{2}x + \frac{9}{2}$.

4 **a** $\frac{dy}{dx} = 4x^3 - 6$. At x = 2, $y = 7$, $\frac{dy}{dx} = 26$. Equation of tangent is thus

$y - 7 = 26(x - 2)$, $y = 26x - 45$ and equation of normal is $y - 7 = -\frac{1}{26}(x - 2)$, giving

$y = -\frac{x}{26} + \frac{92}{13}$.

b $\frac{dy}{dx} = \frac{3}{\sqrt{x}}$. At x = 9, $y = 18$, $\frac{dy}{dx} = 1$. Equation of tangent is thus
$y - 18 = (x - 9)$, $y = x + 9$ and equation of normal is $y - 18 = -(x - 9)$, $y = -x + 27$.

5 Equation of the gradient: $f'(x) = 2x$, $f(2) = f(-2) = 4$, $f'(2) = 4$, $f'(-2) = -4$. Gradients of the normal at x = 2 and −2 will then be $-\frac{1}{4}$ and $\frac{1}{4}$, respectively. Hence, the equations of the normals at x = 2 and −2 are $y - 4 = -\frac{1}{4}(x - 2) \Rightarrow y = -\frac{1}{4}x + \frac{9}{2}$ and $y - 4 = \frac{1}{4}(x + 2) \Rightarrow y = \frac{1}{4}x + \frac{9}{2}$. These curves meet at x = 0, where y = 4.5 hence $(0, 4.5)$.

6 $f'(x) = 2ax + 3$, hence we solve $f'(2) = 4a + 3 = 7$ and $f(2) = 4a + 5 = b$ so that $a = 1, b = 9$.

7 $f'(x) = 2x + k$, hence we solve $f'(1) = 2 + k = 3$ and $f(1) = 1 + k + 3 = b$ so that $k = 1, b = 5$.

8 $f'(x) = 2ax + b$, hence we solve $f'(1) = 2a + b = -1$ (note that the product of the gradient of the tangent and the gradient of the normal is −1) and $f(1) = a + b + 1 = -2$ so that $a = 2, b = -5$.

Exercise 10D

1 The gradient is found using the numerical derivative function on your GDC:

a $\frac{3}{4}$ **b** 2.0986.

2 Using your GDC, plot the curves and find the points at which $\frac{dy}{dx} = 0$:

a Maximum at (−0.786, 13.626) and minimum at (2.120, -23.182)

b Maximum at (0, -1) and minimum at (2, 3)

c no stationary points.

3 **a** $2x = 4, x = 2$ Since $\frac{dy}{dx} = -2$ at $x = 1$ and $\frac{dy}{dx} = 2$ at $x = 3$ the point is a minimum.

b $3x^2 = 12, x = \pm 2$. Since $\frac{dy}{dx} = -12$ at x = 0, and $\frac{dy}{dx} = -15$ at $x = -3$ and at $x = 3$ thenx = −2 is a maximum and x = 2 is a minimum.

c $8 = \frac{2}{x^2}, x = \pm\frac{1}{2}$. Since $\frac{dy}{dx} < 0$ for $-\frac{1}{2} < x < \frac{1}{2}, x \neq 0$ and $\frac{dy}{dx} = 6$ at $x = 1$ and $x = -1$ then $x = -\frac{1}{2}$ is a maximum and $x = \frac{1}{2}$ is a minimum.

4 **a** For each model, find n such that $\frac{dP}{dn} = 0$, calculate $P(n)$ and check if it is a maximum.

As the question does not say differentiation must be used the answers can also be obtained from plotting the curve on a GDC.

i $\frac{dP}{dn} = 0.5 - \frac{4}{(1+n)^2} = 0,\ (1+n)^2 = 8,\ n = -1 \pm \sqrt{8}$, take only the positive solution, $P(\sqrt{8} - 1) = 3.83$. However, $P(0) = 5.5$ and $P(5) = 4.6667$. So the stationary point is a minimum in the interval $n \in [0, 5]$. Thus, $P(0) = 5.5$ gives the maximum profit: 55000 euros for buying 0 parts.

ii $\frac{dP}{dn} = n^2 - 5n + 6 = 0,\ n = 2 \text{ or } 3,\ P(0) = -4, P(2) = \frac{2}{3}, P(3) = \frac{1}{2}$ and $P(5) = 5.1667$, so the maximum profit is 51667 euros, which corresponds to buying 5000 parts.

iii $\frac{dP}{dn} = \frac{n^2}{8} - \frac{5n}{4} + 3 = 0,\ n = 4 \text{ or } 6,\quad P(0) = 0, P(4) = 4.6667, P(5) = 4.5833$, so the stationary point within the interval is a maximum. Maximal profit is 46667 euros for buying 4000 parts.

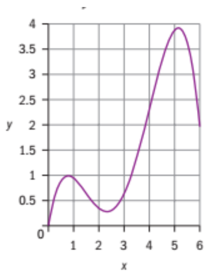
b Hence, they should adopt strategy **i** to maximise their profit.

5 Graph the function on your GDC. Total height John will climb on this route is the height between his starting height and the height of the first maximum and then the height from the minimum to the second maximum.

Maxima at (0.777, 0.986) and (5.150, 3.917)

Minimum at (2.314, 0.279)

Total climb is $0.986+(3.917-0.279)=4.624$ or 462 m.



6 Use your GDC to graph $y = 2^x$ and find its derivative at x = 0, i.e. k = $\ln(2) \approx 0.693$.

Exercise 10E

1 **a** $2(x+l)=100$ metres, so the length $l=50-x$.

b Area $A = x \times l = x(50-x)\text{m}^2$.

c $\frac{dA}{dx} = 50 - 2x$.

d The area is a maximum when $\frac{dA}{dx} = 0,\ x = 25,\ l = 25,\ A = 25 \times 25 = 625\,\text{m}^2$.

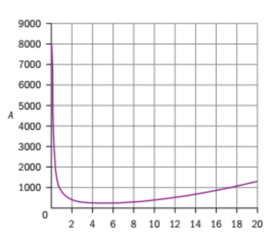
2 **a** The base of the cylinder is a disc of area $A_b = \pi r^2$, and the volume of a cylinder is then $V = A_b h = \pi r^2 h$. s

b $h = \frac{400}{\pi r^2}$.

c Area of an open cylinder = area of the base + curved surface area. The area of the base is πr^2 and the curved surface area is $2\pi rh$. Hence, the total area is $A = \pi r^2 + 2\pi r h$.

d Use the expression for h obtained in part **b** to obtain: $A = \pi r^2 + 2\pi r \times \frac{400}{\pi r^2} = \pi r^2 + \frac{800}{r}$.

e



f Using your GDC find the minimum value on the graph to be $A = 238.53\,\text{cm}^2$ at $r = 5.03$ cm.

3 The total surface area of a closed cylinder is $A = 2\pi r^2 + 2\pi r h = 5000$, where r is its radius and h is its height. We can express the height of the cylinder as $h = \frac{2500}{\pi r} - r$. The volume of the cylinder is $V = \pi r^2 h = 2500r - \pi r^3$. We can find the maximum point exactly in this case by considering its derivative: $\frac{dV}{dr} = 2500 - 3\pi r^2 = 0$ for positive $r = \frac{50}{\sqrt{3\pi}} \approx 16.287$ cm and $h = 32.574$ cm. Thus, $V = \frac{250000}{3\sqrt{3\pi}} \approx 27145$ cm³.

4 **a** Volume of the cone is $V = \frac{1}{3}\pi r^2(18 - r) = 6\pi r^2 - \frac{1}{3}\pi r^3$ so that $\frac{dV}{dr} = 12\pi r - \pi r^2 = \pi r(12 - r)$.

b To find the radius that maximises the volume, solve $\frac{dV}{dr} = 0$ to give $r = 12$ cm and hence $V(12) = 288\pi$ cm³. Using part **a**, when (for example) $x = 10$ $\frac{\mathrm{d}V}{\mathrm{d}r} > 0$ and when $x = 14$ $\frac{\mathrm{d}V}{\mathrm{d}r} < 0$, hence maximum

5 **a** The height of the box is $h = x$, the width is $w = 20 - 2x$ and the length is $l = 24 - 2x$. Hence, the volume is $V = hwl = x(20 - 2x)(24 - 2x) = 4x^3 - 88x^2 + 480x$, $0 \le x \le 10$

b $\frac{dV}{dx} = 12x^2 - 176x + 480$.

c $\frac{dV}{dx} = 0$ for $x = \frac{176 \pm \sqrt{176^2 - 4 \times 12 \times 480}}{24} = 3.62$ or 11.05. Since $x \in (0,10)$, $x = 3.62$ cm.

d At the end points, $V(0) = V(10) = 0$.

e $V(3.62) = 774.16\,\text{cm}^3$.

6 **a** Profit = demand x (price − cost), so $P = \frac{100}{x^2}(x - 0.75) = \frac{100}{x} - \frac{75}{x^2}$.

b $\frac{dP}{dx} = -\frac{100}{x^2} + \frac{150}{x^3}$.

c Solution to $\frac{dP}{dx} = 0$ is

$$\frac{100}{x^2} = \frac{150}{x^3}$$
$$x = \frac{150}{100} = 1.5.$$

d At $x = 1$, $\frac{dP}{dx} = 50$ and at $x = 2$, $\frac{dP}{dx} = \frac{1}{4}(-100 + 75) = -6.25$. Since the derivative of P with respect to x is positive before our stationary point $x = 1.5$ and negative afterwards, P reaches is maximum value at $x = 1.5$.

7 **a** The equation of the circle is $x^2 + y^2 = 36$. If we consider the positive values of x and y then $y = \sqrt{36 - x^2}$. The area of the rectangle is $A = 2x \times 2y = 4x\sqrt{36 - x^2}$ as required.

b Use your GDC to find the value of x that *maximises A*, $x=4.24$cm and $A=72$ cm². This gives $y=\sqrt{36-(4.24...)^2}=4.24...$ so the dimensions of the rectangle are 8.48cm by 8.48cm, and the area is 72cm².

8 **a** Curved surface area of the cylinder $=2\pi\times$ radius of the base $\times$ height of the cylinder, i.e. $A=2\pi\times(8\sin\theta)\times(2\times8\times\cos\theta)=256\pi\sin\theta\cos\theta$ cm^2.

b From the GDC $A=402$ cm² and $\theta=0.785...\left(=\frac{\pi}{4}\right)$

Hence $r=8\sin(0.785)\approx5.66$cm, $h=8\sin(0.785)\approx11.3$cm

Exercise 10F

1 **a** **i** $y=u^3,\ u=x^2+4$

ii $\frac{dy}{du}=3u^2=3(x^2+4),\ \frac{du}{dx}=2x\Rightarrow\frac{dy}{dx}=6x(x^2+4)^2$

b **i** $y=u^2,\ u=5x-7$

ii $\frac{dy}{du}=2u=10x-14,\ \frac{du}{dx}=5\Rightarrow\frac{dy}{dx}=10(5x-7)$

c **i** $y=2u^4,\ u=x^3-3x^2$

ii $\frac{dy}{du}=8u^3=8(x^3-3x^2)^3,\frac{du}{dx}=3x^2-6x\Rightarrow\frac{dy}{dx}=8(3x^2-6x)(x^3-3x^2)^3$

d **i** $y=u^{\frac{1}{2}},\ u=4x-5$

ii $\frac{dy}{du}=\frac{1}{2}u^{-\frac{1}{2}}=\frac{1}{2}(4x-5)^{-\frac{1}{2}},\frac{du}{dx}=4\Rightarrow\frac{dy}{dx}=\frac{2}{(4x-5)^{\frac{1}{2}}}$

e **i** $y=u^{-2},\ u=x^2+1$

ii $\frac{dy}{du}=-2u^{-3}=-2(x^2+1)^{-3},\frac{du}{dx}=2x\Rightarrow\frac{dy}{dx}=-\frac{4x}{(x^2+1)^3}$

f **i** $y=2u^{-\frac{1}{2}},\ u=5x-2$

ii $\frac{dy}{du}=-u^{-\frac{3}{2}}=-(5x-2)^{-\frac{3}{2}},\frac{du}{dx}=5\Rightarrow\frac{dy}{dx}=-\frac{5}{(5x-2)^{\frac{3}{2}}}$

2 **a** $f'(x)=3(x^2+1)^2\times2x=6x^5+12x^3+6x$.

b $g'(x) = 2(5x+2)^{-\frac{2}{3}} \times 5 = \dfrac{10}{(5x+2)^{\frac{2}{3}}}$.

c $h'(x) = 3\left(\sqrt{x}-4\right)^2 \times \dfrac{1}{2\sqrt{x}} = \dfrac{3\left(\sqrt{x}-4\right)^2}{2\sqrt{x}}$.

d $s'(t) = 6\left(t^2-2\right) \times 2t = 12\left(t^3-2t\right)$.

e $v'(t) = -\dfrac{20}{(5t-1)^2}$.

f $a'(t) = 3\left(2-\dfrac{1}{t}\right)^2 \times \dfrac{1}{t^2} = \dfrac{3\left(2-\dfrac{1}{t}\right)^2}{t^2}$.

3 **a** $F'(1) = f'(g(1)) \times g'(1) = f'(3) \times 1 = 5$ **b** $H'(3) = g'(g(3)) \times g'(3) = g'(5) \times 2 = -4$.

Exercise 10G

1 **a** $\dfrac{dy}{dx} = x^2 \times 2 + 2x \times (2x+1) = 2x(3x+1)$.

b $\dfrac{dy}{dx} = (x^2+3) + (x+2) \times 2x = (x^2+3) + 2x(x+2)$

c $\dfrac{dy}{dx} = 3x^2 \times \left(x^2+2x+1\right) + (2x+2) \times \left(x^3-1\right) = (x+1)\left(5x^3+3x^2-2\right)$.

d $s'(t) = \left(2t^2+3\right) \times 5(4-t)^4 + 4t \times (4-t)^5 = (4-t)^4\left(-14t^2+16t-15\right)$.

e $f'(x) = \dfrac{1}{x}\left(6x^2+1\right) - \dfrac{1}{x^2}\left(2x^3+x+4\right) = \dfrac{4\left(x^3-1\right)}{x^2}$.

f $g'(t) = (2t+1)^4 \times 6t^2\left(t^3+1\right) + 8(2t+1)^3\left(t^3+1\right)^2 = 2(2t+1)^3\left(t^3+1\right)\left(10t^3+3t^2+4\right)$.

2 **a** $\dfrac{dy}{dx} = \dfrac{\left(6x^2+1\right)x - (2x^3+x+4)}{x^2} = \dfrac{4x^3-4}{x^2}$

b $\dfrac{dy}{dx} = \dfrac{-\left(x^2+1\right) - 2x(1-x)}{(x^2+1)^2} = \dfrac{x^2-2x-1}{(x^2+1)^2}$.

c $\dfrac{dy}{dx} = \dfrac{\sqrt{x+1} - \dfrac{x}{2\sqrt{x+1}}}{x+1} = \dfrac{2x+2-x}{2(x+1)^{\frac{3}{2}}} = \dfrac{x+2}{2(x+1)^{\frac{3}{2}}}$.

d $\dfrac{ds}{dt} = \dfrac{4(2t+1)-8t}{(2t+1)^2} = \dfrac{4}{(2t+1)^2}$.

e $f'(x) = \dfrac{3x^2(x+1) - \left(x^3-1\right)}{(x+1)^2} = \dfrac{2x^3+3x^2+1}{(x+1)^2}$.

f $g'(t)=\dfrac{\frac{1}{3}t^{-\frac{2}{3}}\left(t^{\frac{1}{3}}-4\right)-\frac{1}{3}t^{-\frac{2}{3}}t^{\frac{1}{3}}}{\left(t^{\frac{1}{3}}-4\right)^2}=-\dfrac{4}{3t^{\frac{2}{3}}\left(t^{\frac{1}{3}}-4\right)^2}$.

3 **a** $\dfrac{dy}{dx}=2x\left(x^2+1\right)^3+3\left(x^2+1\right)^2\times 2x\times x^2=2\left(x^2+1\right)^2x\left(4x^2+1\right)$.

b $\dfrac{dy}{dx}=-\dfrac{8}{(2x+3)^2}$.

c $\dfrac{dy}{dx}=\dfrac{\sqrt{2x+1}-\frac{x}{\sqrt{2x+1}}}{2x+1}=\dfrac{x+1}{(2x+1)^{\frac{3}{2}}}$.

d $\dfrac{ds}{dt}=4\sqrt{2t-3}+\dfrac{4t}{\sqrt{2t-3}}=\dfrac{12(t-1)}{\sqrt{2t-3}}$.

e $f'(x)=\dfrac{(3-2x)^2-x\times 2(3-2x)\times(-2)}{(3-2x)^4}=\dfrac{(3-2x)(3-2x+4x)}{(3-2x)^4}=\dfrac{3+2x}{(3-2x)^3}$.

f $g'(t)=\dfrac{\left(4(t+1)^2+4t\times 2(t+1)\right)(2t-3)-2\times 4t(t+1)^2}{(2t-3)^2}=\dfrac{4(t+1)\left(4t^2-9t-3\right)}{(2t-3)^2}$.

4 **a** $\dfrac{dy}{dx}=(2x-1)^2+x\times 4(2x-1)=(2x-1)(6x-1)$.

b **i** Gradient of the tangent at P is $(2-1)(6-1)=5$, and the equation of the tangent is thus $y=5x-4$.

ii Gradient of the normal at P is $-\frac{1}{5}$ and the equation of the normal is thus:

$y=-\frac{1}{5}x+\frac{6}{5}$.

c Height of the triangle is 1 (y coordinate at point P). The tangent line crosses x axis at $x=\frac{4}{5}$ and the normal line crosses the x axis at $x=6$ so the base of the triangle is of length 5.2. Area of the triangle is thus $\frac{1}{2}\times 1\times 5.2=2.6$.

5 **a** First, find $\dfrac{dy}{dx}=\dfrac{4(x+1)-(4x-2)}{(x+1)^2}=\dfrac{6}{(x+1)^2}$. The gradient of the normal at $x=2$ is then $-\dfrac{(2+1)^2}{6}=-1.5$. The equation of the normal is $y=-1.5x+5$ and it goes through the point $(2,2)$.

b Equate the original equation to the equation of the normal:

$$\frac{4x-2}{x+1} = 5-1.5x,$$
$$4x-2 = -1.5x^2+3.5x+5,$$
$$-1.5x^2-0.5x+7=0,$$
$$x = -\frac{7}{3} \text{ or } 2.$$

At $x=-\frac{7}{3}, y = \dfrac{4\times\left(-\frac{7}{3}\right)-2}{-\frac{7}{3}+1} = \dfrac{-28-6}{-7+3} = \dfrac{34}{4} = 8.5.$

Point is (-7/3, 8.5)

6 $\dfrac{dy}{dx} = -u\dfrac{dv}{dx}\dfrac{1}{v^2} + \dfrac{du}{dx}\dfrac{1}{v} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}.$

Exercise 10H

1 **a** $\dfrac{dy}{dx} = 4\cos x.$

b $\dfrac{dy}{dx} = -2\sin x - 2x.$

c $\dfrac{dy}{dx} = \dfrac{5}{\cos^2 x}.$

d Use chain rule: $\dfrac{dy}{dx} = -8\sin 4t.$

e $f'(x) = 5\cos 5x + 12x^2.$

f $g'(x) = 3\cos 3t + 6\sin 2t - 2t.$

2 **a** $\dfrac{dy}{dx} = x\cos x + \sin x.$

b $\dfrac{dy}{dx} = -\sin^2 x + \cos^2 x$

c $\dfrac{dy}{dx} = \dfrac{2x^2}{\cos^2 x} + 4x\tan x.$

d $\dfrac{dy}{dt} = \dfrac{t\cos t - \sin t}{t^2}.$

e $f'(x) = \dfrac{-\sin^2 x - \cos^2 x}{\sin^2 x} = -\dfrac{1}{\sin^2 x}.$

f $g'(t) = \dfrac{2\cos 2t\cos t + \sin 2t\sin t}{\cos^2 t}$

3 **a** $\dfrac{dy}{dx} = 2x\cos x^2.$

b $\dfrac{dy}{dx} = -6\sin 3x.$

c $\dfrac{dy}{dx} = \dfrac{3}{\cos^2(3x-1)}.$

d $\dfrac{dy}{dt} = 2\sin t\cos t.$

e $f'(x) = -9\cos^2 x\sin x.$

f $g'(t) = -8\sin^2 4t + 8\cos^2 4t = 8\left(\cos^2 4t - \sin^2 4t\right)$

g $f'(x) = 2\sin 3x\cos 3x \times 3 = 6\sin 3x\cos 3x.$

h $g'(t) = -\sin^2 t\sin t + \cos t\, 2\sin t\cos t$
$= -\sin^3 t + 2\sin t\cos^2 t$

4 **a** Area of the screen $A = 2xy = 10.2x\cos\dfrac{\pi}{10}x$, since the screen touched the curve y.

b $\frac{dA}{dx} = -1.02\pi x \sin\frac{\pi}{10}x + 10.2\cos\frac{\pi}{10}x.$

c Use your GDC to obtain the maximum value $A(2.74) = 18.22.$

Exercise 10I

1 a $\frac{dy}{dx} = \frac{2x}{x^2+1}.$

b $\frac{dy}{dx} = xe^x + e^x = e^x(1+x).$

c $\frac{dy}{dx} = 4xe^{2x^2}.$

d $\frac{dy}{dt} = \frac{t^2}{t} + 2t\ln t = t(1+2\ln t).$

e $f'(x) = (2x+1)2e^{2x} + 2e^{2x} = 4e^{2x}(x+1).$

f $f(t) = \frac{\frac{1}{t}t - \ln t}{t^2} = \frac{1-\ln t}{t^2}.$

g $s'(t) = \frac{3t}{t^2-2}\times 2t + 3\ln(t^2-2)$
$= \frac{6t^2}{t^2-2} + 3\ln(t^2-2)$

h $g'(x) = \frac{8e^{4x}x - 2e^{4x}}{x^2} = \frac{2e^{4x}}{x^2}(4x-1).$

i $h'(t) = t\times 6te^{3t^2} + e^{3t^2} = e^{3t^2}(1+6t^2).$

2 a $\frac{dy}{dx} = (5\ln x)' = \frac{5}{x}.$

b $\frac{dy}{dx} = (4\ln(2x+3))' = \frac{8}{2x+3}.$

c $\frac{dy}{dx} = (\ln x + 2\ln(x-3))' = \frac{1}{x} + \frac{2}{x-3}.$

d $\frac{dy}{dx} = (\ln(2x+1) - \ln x)' = \frac{2}{2x+1} - \frac{1}{x}.$

e $\frac{dy}{dx} = (x^2\ln e)' = 2x.$

f $\frac{dy}{dx} = (2\ln x - 3\ln(2x+1))' = \frac{2}{x} - \frac{6}{2x+1}.$

3 a $(\ln ax)' = (\ln a + \ln x)' = \frac{1}{x}.$

b $(\ln ax)' = \frac{1}{ax}\times a = \frac{1}{x}.$

4 a $f'(x) = 20\cos(\ln x)\times\frac{1}{x}$, using GDC $f'(x) = f(x)$ at $x = 0.175.$

b $f'(x) = \frac{1}{\cos x + 2}\times(-\sin x)$, using GDC $f'(x) = f(x)$ at $x = 3.14\ldots$ $(\pi).$

c $f'(x) = 2\cos xe^{2\sin x}, f'(x) = f(x)$ when $2\cos x = 1,\ x = 1.05\ldots\left(\frac{\pi}{3}\right).$

5 a Tangent to the curve has gradient $\frac{dy}{dx} = \frac{2e^{2x}(x+3) - e^{2x}}{(x+3)^2} = \frac{e^{2x}(2x+5)}{(x+3)^2}.$

Parallel to the x-axis when $\frac{dy}{dx} = 0$, i.e. when $x = -2.5$ and $y = 2e^{-5}.$

b Normal is parallel to the y axis, so $x = -2.5$.

Exercise 10J

1 **a** $\frac{dy}{dx} = x\cos x + \sin x$, $\frac{d^2y}{dx^2} = -x\sin x + 2\cos x$.

b $f'(x) = -\frac{4}{(x-3)^3}, f''(x) = \frac{12}{(x-3)^4}$.

c $\frac{ds}{dt} = \frac{2t}{t} + 2\ln t = 2(1+\ln t)$, $\frac{d^2s}{dt^2} = \frac{2}{t}$.

2 **a** $\frac{dy}{dx} = 5x^4 - 6x = 0$ at $x = 0$, $x^3 = \frac{6}{5}, x = \left(\frac{6}{5}\right)^{\frac{1}{3}} \approx 1.06$.

When $x = 0$, $y = 5$ and when $x = 1.06$, $y = 2.97$

To distinguish between minimum and maximum points, consider $\frac{d^2y}{dx^2} = 20x^3 - 6$.

At $x = 0, \frac{d^2y}{dx^2} = -6 < 0$, hence the point is a maximum.

At $x = 1.06\ldots$, $\frac{d^2y}{dx^2} = 18 > 0$, hence the point is a minimum.

b Inflexion point occurs when $\frac{d^2y}{dx^2} = 0$, $x = 0.3^{\frac{1}{3}} \approx 0.669$. $y \approx 3.79$

c The curve is thus concave up for $x > 0.669$

3 **a** **i** $\frac{dy}{dx} = \frac{xe^x - e^x}{x^2} = \frac{e^x(x-1)}{x^2}$

ii $\frac{d^2y}{dx^2} = \frac{x^2\left(e^x(x-1) + e^x\right) - \left(e^x(x-1)2x\right)}{x^4} = \frac{e^x\left(x^2 - 2x + 2\right)}{x^3}$

b At $x = 1$, $\frac{dy}{dx} = 0, \frac{d^2y}{dx^2} = e > 0$, so it is a minimum point.

c Points of inflection would occur if $\frac{d^2y}{dx^2} = 0$, i.e. if $\frac{e^x\left(x^2 - 2x + 2\right)}{x^3} = 0 \Rightarrow x^2 - 2x + 2 = 0$. The discriminant is $4 - 4 \times 2 = -4 < 0$ so there are no real roots of this equation and hence no points of inflection exist.

4 **a** $f'(x) = \frac{(x-3) - x}{(x-3)^2} = -\frac{3}{(x-3)^2} < 0$ for all values of x.

b $f''(x) = \frac{6}{(x-3)^3}$.

For $x < 3$, then $f''(x) < 0$.

For $x = 3$, then $f''(x)$ is infinite.

For $x > 3$, then $f''(x) > 0$.

Hence, $f''(x) \neq 0$ for all x.

5 **a** **i** $\frac{dy}{dx} = 3(x-2)^2(x+1) + (x-2)^3 = (x-2)^2(4x+1)$

From a graph of the function, there is a local minimum at $x = -\frac{1}{4}$ with $y = -8.54$.

There is no local maximum.

ii $\frac{d^2y}{dx^2} = 2(x-2)(4x+1) + 4(x-2)^2$

$= 6(x-2)(2x-1)$

iii $\frac{d^2y}{dx^2} = 0$ for $x = 2$ or $\frac{1}{2}$. The points of inflection are $(2,0)$ and $(0.5,-5.0625)$.

iv The function is concave up when x<0.5, x>2 and concave down when 0.5<x<2

b **i** $h'(x) = e^{-x}(-\cos x - \sin x) = 0$ for $x = 2.356\ldots\left(\frac{3\pi}{4}\right)$ or $5.4977\ldots\left(\frac{7\pi}{4}\right)$.

From a graph of the function, there is a minimum at $(2.36,-0.067)$ and a maximum at $(5.50,0.00290)$.

ii $h''(x) = 2e^{-x}\sin x$

iii $h''(x) = 0$ for $x = 0, \pi$ or 2π. The inflection points are $(0,1)$, $(3.14,-0.0432)$ and $(6.28,0.00187)$.

iv The function is concave up for $0 < x < \pi$ and concave down for $\pi < x < 2\pi$

c **i** $f'(x) = e^{-2x}(-2x^3 + 3x^2) = x^2e^{-2x}(3-2x) = 0$ for $x = 0$ or $\frac{3}{2}$.

From a graph of the function, there is a maximum at $(1.5, 0.168)$. There is no minimum.

ii $f''(x) = e^{-2x}2x(2x^2 - 6x + 3)$

iii $f''(x) = 0$ for $x = 0$ and for $x = \frac{6 \pm \sqrt{36-24}}{4} = \frac{1}{2}(3 \pm \sqrt{3}) = 2.366\ldots, 0.6339\ldots$. The inflection points are $(0,0)$, $(2.37, 0.117)$ and $(0.634, 0.0717)$.

iv The curve is concave down for x<0, 0.634<x<2.37, concave up when 0<x<0.634, x> 2.37 from sign of second derivative.

6 **a** **i** $p'(t) = \frac{-10}{(1+4e^{-2t})^2} \times (-8e^{-2t}) = \frac{80e^{-2t}}{(1+4e^{-2t})^2}$.

ii Since both numerator and denominator are strictly positive (e^{-2t} never reaches 0) for all t, $p'(t) > 0$. Hence, the population is always growing.

b **i** $p''(t) = \dfrac{80\left(-2e^{-2t}\left(1+4e^{-2t}\right) + 16e^{-4t}\right)}{\left(1+4e^{-2t}\right)^3} = \dfrac{160e^{-2t}\left(4e^{-2t}-1\right)}{\left(1+4e^{-2t}\right)^3}$

ii $p''(t) = 0$ when

$$4e^{-2t} = 1$$
$$e^{-2t} = \frac{1}{4}$$
$$-2t = \ln\frac{1}{4} = -2\ln 2$$
$$t = \ln 2 \approx 0.693$$

iii This is a point of inflection, which is when the rate of population growth is a maximum. After it, the population starts growing at a decreasing rate.

c **i** $p(\ln 2) = \dfrac{10}{1+4\times 2^{-2}} = 5.$

ii This is half of the carrying capacity. It is half of the maximum population that the environment can sustain indefinitely.

Exercise 10K

1 **a** Average velocity: $\dfrac{h(1.5)-h(1)}{1.5-1} = \dfrac{9-8}{0.5} = 2$ m/s

b Instantaneous velocity: $h'(t) = 12-8t, h'(1) = 4$ m/s

2 **a** Average velocity: $\dfrac{s(3)-s(0)}{3} = \dfrac{1-(-2)}{3} = 1$ m/s

b Instantaneous velocity: $s'(t) = -3t^2+4t+4$ so $s'(3) = -11$ m/s. Instantaneous acceleration: $s''(t) = -6t+4$ so $s''(3) = -14$ m/s^2

c At $t = 3$ s, the velocity of the particle is negative and the acceleration is negative. Hence, the speed of the particle (absolute value of the velocity) is increasing.

d The direction of the particle changes when its velocity changes sign. $s'(t) = 0$ at $t = 2$ and at $-\dfrac{2}{3}$ s. For positive times, the direction of the particle changes at $t = 2$ s.

e **i**

$$s''(t) = 0$$
$$6t = 4$$
$$t = \frac{2}{3} \text{ s}$$

ii $s''\left(\dfrac{2}{3}\right) = 0$ but $s'\left(\dfrac{2}{3}\right) = \dfrac{16}{3}$ so this is a point of inflection.

3 Particle's acceleration is $a = 5\sin 3x \times 15\cos 3x$. At $x = 2$, the acceleration becomes $a = 5\sin 6 \times 15\cos 6 \approx -20.1$.

4 **a** Velocity of the rocket: $v(t) = y'(t) = -0.4t + 2$.

b **i** Initial velocity of the rocket: $v(0) = 2$ m/s

ii Maximum height of the rocket occurs when $v(t) = 0$ giving $t = 5$. Thus,

$y = -0.2(5)^2 + 2(5) + 1 = 6\,\text{m}$.

iii The rocket hits the ground when $y(t) = 0$, which occurs for $t = 5 \pm \sqrt{30}$. Since $t \geq 0$,

$v\left(5 + \sqrt{30}\right) \approx -2.19$ m/s.

5 **a** **i** $s(2) = 0.4\left(2 + 2 - 2e^{-1}\right) \approx 1.31$ m

ii To solve for time such that $2 = 0.4\left(2 + t - 2e^{-0.5t}\right)$, graph $2 - 0.4\left(2 + t - 2e^{-0.5t}\right)$ on your GDC and find the zero of the function to be at $t \approx 3.37$ s.

b Velocity of the marble: $s'(t) = 0.4\left(1 + e^{-0.5t}\right)$.

c $s'(0) = 0.8$ m/s

d As $t \to \infty, s'(t) \to 0.4$ m/s

e The marble is first moving within 1% of its terminal velocity when $e^{-0.5t} = 0.01$, giving $t = -2\log 0.01 = \log 100^2 \approx 9.21$ s

6 **a** Solving $200 = 3t + \ln(2t + 1)$ for time using a graph of $200 - 3t + \ln(2t + 1) = 0$ on your GDC, then the zero of the function is $t \approx 65.04$ s.

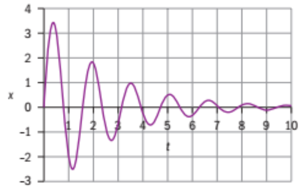
b Velocity of the cyclist: $v(t) = \dfrac{dx}{dt} = 3 + \dfrac{2}{2t + 1}$.

c *i* Initial velocity of the cyclist: $v(0) = 3 + 2 = 5\,\text{ms}^{-1}$.

ii At the top of the hill, $v(65.04) = 3.02\ \text{ms}^{-1}$.

d Acceleration: $a(t) = v'(t) = -\dfrac{4}{(2t + 1)^2} < 0 \quad \forall t$.

7 **a** Use your GDC to sketch the curve:



b Use the graph to determine the time t and note that the greatest such value of t will correspond to a negative displacement: $-1.2 = 4e^{-0.4t}\sin 4t$ for $t = 2.84$ s.

c The velocity of the weight $x(t) = \dfrac{dx}{dt} = 4e^{-0.4t}\left(-0.4\sin 4t + 4\cos 4t\right)$.

d **i** Weight first returns to $x = 0$ when

$$4t = \pi$$

$$t = \frac{\pi}{4}$$

This can be found directly from the GDC as 0.785 s

ii $\dot{x}(0.785) = -11.69\,\text{cm/s}$

Exercise 10L

1 Volume of a cylinder is given by $V = \pi r^2 h$.

a $\frac{dV}{dt} = \pi r^2 \frac{dh}{dt}$.

b $\frac{dV}{dt} = 2\pi r \frac{dr}{dt} h$.

2 The volume of a cube with a side l is $V = l^3$. For $27 = l^3$ we have $l = 3$. The rate of change of the volume $\frac{dV}{dt} = 3l^2 \frac{dl}{dt} = 2\ \text{m}^3\text{s}^{-1}$, hence $\frac{dl}{dt} = \frac{2}{27}$ ms^{-1}. The rate of change of the surface area ($A = 6l^2$) is $\frac{dA}{dt} = 12l \frac{dl}{dt} = 12 \times 3 \times \frac{2}{27} = \frac{8}{3}$ m^{2}s^{-1}.

3 Let the horizontal distance between the wall and the bottom of the ladder be s and the angle between the ladder and the wall be α. Then, $\frac{s}{4} = \sin\alpha$ so $\frac{ds}{dt} = 4\cos\alpha \frac{d\alpha}{dt} = 1.5$. Hence, when $\alpha = \frac{\pi}{3}, \frac{d\alpha}{dt} = \frac{1}{4} \times \frac{1.5}{\cos\frac{\pi}{3}} = 0.75\,\text{rad s}^{-1}$.

4 Let h be the height of the water. The cross-section forms a triangle which is similar to the cross-section of the trough, so the sides are in the same ratio. If the width of the triange with height h is b then $\frac{b}{h} = \frac{1}{1.2} \Rightarrow b = \frac{h}{1.2}$

Let the volume of water in cm^3 at time t be V where t is time in minutes. When the height of the water is h, the volume $V = \frac{1}{2}hbl = \frac{1}{2} \times h \times \frac{h}{1.2} \times 400 = \frac{500h^2}{3}$ so $\frac{dV}{dh} = \frac{1000h}{3}$ and $\frac{dV}{dt} = 1200$. Thus, $\frac{dh}{dt} = \frac{dV}{dt} \div \frac{dV}{dh} = 1200 \div \frac{1000h}{3} = \frac{3.6}{h}$. When $h = 45\,\text{cm}$, $\frac{dh}{dt} = \frac{3.6}{45} = 0.08$ cm/min.

5 Area of sector is $\frac{1}{2}r^2\theta$

Area of triangle is $\frac{1}{2}r^2 \sin\theta$

Area shaded is $A = \frac{1}{2}\theta r^2 - \frac{1}{2}r^2 \sin\theta = \frac{25}{2}(\theta - \sin\theta)$.

Since $\frac{dA}{d\theta}=\frac{25}{2}(1-\cos\theta)$, then, $\frac{dA}{dt}=12.5(1-\cos\theta)\frac{d\theta}{dt}=2.5(1-\cos 1)=1.15\,\text{cm}^2\text{s}^{-1}$.

6 a $f(x)=\frac{\cos x}{\sin x}, f'(x)=\frac{-\sin x\times\sin x-\cos x\times\cos x}{\sin^2 x}=-\frac{1}{\sin^2 x}$.

b The horizontal distance between the observer and the plane is $s=\frac{2000}{\tan\theta}$ m. Since $\frac{ds}{d\theta}=-\frac{2000}{\sin^2\theta}$ and $\frac{d\theta}{dt}=0.01$ for $\theta=0.5$ then $\frac{ds}{dt}=-\frac{2000}{\sin^2 0.5}0.01=-87.01\,\text{ms}^{-1}$ so the plane is travelling at 87.01 m s^{-1}.

Chapter review

1 a $\frac{dy}{dx}=\frac{3}{2}+8x$, at $x=4$, gradient is 33.5.

b $f'(x)=5x^{\frac{1}{4}}-80=0$ when $x=16^4=65536$, $f(65536)=-1048584$. Point is (65536, -1 048 584)

c $\frac{dr}{dt}=10-14t$, at $t=5$, then $\frac{dr}{dt}=-60$.

2 a $f'(x)=-2\sin x+2\cos 2x$.

b $f'(x)=\frac{1}{\tan x}\times\frac{1}{\cos^2 x}=\frac{2}{\sin 2x}$.

c $\frac{dy}{dx}=-\frac{1}{\sin^2 x}$.

d $\frac{dy}{dx}=e^{2x}(2x+1)$.

3 a $f'(x)=-\frac{24}{x^3}+\frac{3}{2}x^{-\frac{1}{2}}$, $f''(x)=\frac{72}{x^4}-\frac{3}{4}x^{-\frac{3}{2}}$.

b $f'(x)=(3x-7)+3(x-1)=6x-10, f''(x)=6$.

c $\frac{dy}{dx}=\frac{x^2-2x(x-12)}{x^4}=\frac{-x^2+24x}{x^4}=\frac{24-x}{x^3}, \frac{d^2y}{dx^2}=\frac{-x^3-3x^2(24-x)}{x^6}=\frac{2x-72}{x^4}$.

d $\frac{dy}{dx}=\frac{2x(x-12)-x^2}{(x-12)^2}=\frac{x^2-24x}{(x-12)^2}, \frac{d^2y}{dx^2}=\frac{(2x-24)(x-12)^2-2(x-12)(x^2-24x)}{(x-12)^4}=\frac{288}{(x-12)^3}$.

4 $f'(x)=\frac{-2e^{2x}}{(e^{2x}+1)^2}<0$ so function is decreasing for all x.

5 The gradient of the tangent to the curve is $\frac{dy}{dx}=6x-7$, and the gradient of the normal to the curve is $-\frac{1}{\frac{dy}{dx}}=\frac{1}{7-6x}$. Hence, the equation of the normal to the curve at $(2,6)$ is $y=-\frac{x}{5}+6\frac{2}{5}$ while the equation of the normal to the curve at $(0,8)$ is $y=\frac{x}{7}+8$. They intersect when $-\frac{x}{5}+6\frac{2}{5}=\frac{x}{7}+8, x=-\frac{14}{3}$ so the point of intersection is $\left(-\frac{14}{3},\frac{22}{3}\right)$.

a $\frac{dy}{dx} = 7\left(1 - x^{-\frac{1}{2}}\right) = 0$ at $x = 1$, so coordinates are $(1, -7)$

b $\frac{d^2y}{dx^2} = \frac{7}{2}x^{-\frac{3}{2}}$

c Hence, the point is a minimum as $\frac{d^2y}{dx^2} > 0$.

6 **a** The area of the enclosure: $A = x \times (128 - 2x) = 2x(64 - x)$.

b The area is maximum when $\frac{dA}{dx} = 128 - 4x = 0, x = 32$ m

c The maximum area of the enclosure $A(32) = 32 \times 64 = 2048$ m^2

7 **a** $v(2) = 3\,\text{ms}^{-1} > 0$ so the particle is moving to the right.

b **i** $0 \le t < 7$

ii $t > 7$

Particle is moving to the right when $v(t) > 0$ and to the left when $v(t) < 0$.

c At $t = 3$, the velocity is increasing so the acceleration is positive.

d

	Direction of motion	Velocity	Acceleration
[0,4[	Right	Increasing	Positive
]4,6[	Right	Constant, 4 m/s	Zero
]6,7[	Right	Decreasing	Negative
]7,8.5[	Left	Decreasing	Negative
]8.5,10[	Left	Increasing	Positive

e The particle is farthest to the right at $t = 7$ s, because it is travelling right until then. After $t = 7$ s, it is travelling left.

8 The gradient $f'(x) = \frac{140e^{-2x+19}}{\left(1 + e^{-2x+19}\right)^2}$ has local maximum when

$f''(x) = \frac{-280e^{2x}\left(e^{2x+19} - e^{38}\right)}{\left(e^{2x} + e^{19}\right)^3} = 0, e^{2x} = e^{19}$ giving $x = 9.5$. Since $f'(9.5) = 35$, then (9.5, 35) is a point of inflection for $f(x)$.

$f(x-5) = \frac{70}{1 + e^{-2(x-5)+19}} = \frac{70}{1 + e^{-2x+29}}$. Hence, we can infer that the point of inflection $f(x-5)$ occurs for $-2x = 29$ or $x = 14.5$. Thus, it is (14.5, 35).

9 The distance between the origin and the particle is $s = \sqrt{x^2 + y^2} = \sqrt{x^2 + x + 6\sqrt{x} + 9}$.

So, $\frac{ds}{dx} = \frac{2x+1+\frac{3}{\sqrt{x}}}{2\sqrt{x^2+x+6\sqrt{x}+9}}$, giving

$$\frac{ds}{dt} = \frac{ds}{dx}\frac{dx}{dt} = \frac{2x+1+\frac{3}{\sqrt{x}}}{2\sqrt{x^2+x+6\sqrt{x}+9}} \times 2 = \frac{2x+1+\frac{3}{\sqrt{x}}}{\sqrt{x^2+x+6\sqrt{x}+9}}.$$

When $x=1$, then

$$\frac{ds}{dt} = \frac{2+1+\frac{3}{1}}{\sqrt{1+1+6+9}} = \frac{6}{\sqrt{17}} \text{ cm/min.}$$

Exam style questions

10 The gradient of the tangent to the curve is $\frac{dy}{dx} = -2x^3 = -2$. The gradient of the normal to the curve is thus $\frac{1}{2}$. The y coordinate at x = 1 is $y = \frac{3}{2}$. The equation of the normal is thus $y = \frac{x}{2} + 1$.

11 a The gradient of the curve is $\frac{dy}{dx} = 2x^2 - 7x + 2 = -3$. Solve $2x^2 - 7x + 5 = 0$, so that $x = \frac{1}{4}\left(7 \pm \sqrt{49-40}\right) = 2.5$ or 1. Hence the coordinates of these points are $\left(1, \frac{25}{6}\right)$ and $\left(\frac{5}{2}, -\frac{35}{24}\right)$.

b Solve $2x^2 - 7x + 2 = 0$, $x = \frac{1}{4}\left(7 \pm \sqrt{33}\right) = 0.3138...$ or $3.186...$ The gradient is negative between these values.

12 a The height of the box is $h = x$, the width is $w = 30 - 2x$ and the length is $l = 40 - 2x$. Hence, the volume is $V = hwl = x(30-2x)(40-2x) = 4x^3 - 140x^2 + 1200x$.

b $\frac{dV}{dx} = 12x^2 - 280x + 1200$.

c $\frac{dV}{dx} = 0$ for $12x^2 - 280x + 1200 = 0, x^2 - \frac{70}{3}x + 100 = 0$.

d Using your GDC, plot $V(x)$ and find the maximum value in the range $x \in (0,15)$ $V(5.66) = 3032.3 \text{ cm}^3$.

13 a $f'(x) = \frac{2x(2x^3-1) - 6x^4}{(2x^3-1)^2} = \frac{-2x^4 - 2x}{(2x^3-1)^2} = \frac{-2x(x^3+1)}{(2x^3-1)^2}$

b $f(1) = 1$ and $f'(1) = -4$, so the equation of the tangent at the point (1,1) is $y = -4x + 5$.

c Gradient is zero when $x\left(1+x^3\right)=0$ giving $x=0$ or $x=-1$ so the coordinates are (0,0) and $\left(-1,\frac{-1}{3}\right)$

d $f(x)$ is increasing for positive gradient. This happens for $-1<x<0$.

14 Volume of a cone is given by $V=\frac{1}{3}\pi r^2 h$. Since the height is fixed, $\frac{dV}{dt}=\frac{2}{3}\pi r\frac{dr}{dt}h$ and

$$\frac{dr}{dt}=\frac{dV}{dt}\frac{3}{2}\frac{1}{\pi rh}=2\times\frac{3}{2}\times\frac{1}{\pi\times 0.4\times 50}=\frac{3}{20\pi}\text{ cm/min}$$

15 a $x(2)=\frac{2}{-3}+2\ln 3=1.53\text{ m}$. (Need to check if the displacement had not yet started decreasing at $t=2$ as in that case the distance travelled would not be equal to the displacement. $x'(t)=0$ at $t=2.18$. as shown in part **b**.)

b $\frac{dx}{dt}=-\frac{5}{(t-5)^2}+\frac{2}{1+t}=0$ when

$$5+5t=2t^2-20t+50$$
$$2t^2-25t+45=0$$
$$t=\frac{1}{4}\left(25\pm\sqrt{256}\right).$$

In the given interval, the particle is stationary at $t=\frac{1}{4}\left(25-\sqrt{256}\right)$ s.

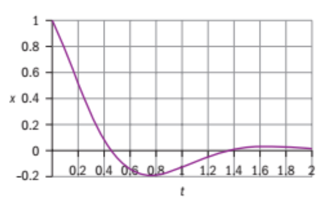
16 a $\frac{dx}{dt}=e^{-2t}\left(-2\cos\sqrt{12}t-\sqrt{12}\sin\sqrt{12}t\right)$

$$\frac{d^2x}{dt^2}=e^{-2t}\left(4\cos\sqrt{12}t+2\sqrt{12}\sin\sqrt{12}t+2\sqrt{12}\sin\sqrt{12}t-12\cos\sqrt{12}t\right)$$
$$=e^{-2t}\left(-8\cos\sqrt{12}t+4\sqrt{12}\sin\sqrt{12}t\right).$$

Substitute into the given equation:

$$e^{-2t}\left(-8\cos\sqrt{12}t+4\sqrt{12}\sin\sqrt{12}t-8\cos\sqrt{12}t-4\sqrt{12}\sin\sqrt{12}t+16\cos\sqrt{12}t\right)=0.$$

b



c The bearing comes to rest when $\frac{dx}{dt}=0$. Using the expression found in part **a**,

$$-2\cos 2\sqrt{3}t-2\sqrt{3}\sin 2\sqrt{3}t=0$$
$$\tan 2\sqrt{3}t=-\frac{1}{\sqrt{3}}$$

Hence,

$$2\sqrt{3}t = \frac{5\pi}{6} + n\pi$$

From the graph, we want the first stationary point, so

$$t = \frac{5\pi}{12\sqrt{3}} \approx 0.756\,\text{s}$$

$$x = -0.191 \text{ m}$$

d Maximum speed is maximum absolute value of the velocity, which will occur for $a = \frac{d^2x}{dt^2} = 0$.

Using the expression found in part **a**,

$$\tan 2\sqrt{3}t = \frac{1}{\sqrt{3}}$$

$$2\sqrt{3}t = \frac{\pi}{6} + n\pi$$

For $n = 0$,

$$t = \frac{\pi}{12\sqrt{3}} \approx 0.151 \text{ s}$$

At this time, $\frac{dx}{dt} = -2.56$ so the maximum speed is 2.56 m/s and $x(0.15) = 0.641$ m.

11 Approximating irregular spaces: integration and differential equations

Skills check

1 $A = \frac{1}{2} \times 1.8(3.2 + 4.8) = 7.2\ \text{cm}^2$.

2 **a** $\frac{dy}{dx} = 9x^2 - \frac{1}{\sqrt{x}} - \frac{8}{x^3}$ or $= 9x^2 - x^{-\frac{1}{2}} - 8x^{-3}$ **b** $f'(x) = -5\sin 5x + 2\sin x \cos x$

c $\frac{ds}{dt} = \frac{1}{t} - 4te^{t^2}$.

Exercise 11A

1 **a** **i** $y = \sqrt{9 - x^2}, x \in [0, 3]$ **ii** $A = \int_0^3 \sqrt{9 - x^2}\, dx$

b

x	0	0.5	1.0	1.5	2.0	2.5	3.0
y	3.000	2.958	2.828	2.598	2.236	1.658	0.000

c **i** $A_l = R1 + R2 + R3 + R4 + R5 + R6 = 6.139$ **ii** $A_u = R1 + R2 + R3 + R4 + R5 + R6 = 7.639$.

d $\bar{A} = 6.889$

e Actual area is $A = \frac{1}{4}\pi r^2 = 7.1$, percentage error $= 3\%$.

2 **a**

b $R = \int_1^6 \frac{1}{x}\, dx$.

c Use equally spaced rectangles with vertices at:

x	1.00	2.00	3.00	4.00	5.00	6.00
y	1.000	0.500	0.333	0.250	0.200	0.167

i $A_l = 1.450$ **ii** $A_u = 2.283$.

d $1.450 < R < 2.283$.

Exercise 11B

1 **a** $A = \int_2^4 x^2 dx = \frac{56}{3}$ **b** $A = \int_{-1}^1 2^x dx = 2.16$

c $A = \int_{-1}^{1} \frac{1}{1+x^2}dx = 1.57$

d $A = \int_{0.5}^{3} \frac{1}{x}dx = 1.79$

e $A = \int_{0}^{1} -(x-3)(x+2)\,dx = \frac{37}{6} = 6.17$

f $A = \int_{-2}^{0} -(x-3)(x+2)\,dx = \frac{22}{3} = 7.33$ or $A = \int_{0}^{3} -(x-3)(x+2)\,dx = \frac{27}{2} = 13.5$

g $A = \int_{-2}^{3} -(x-3)(x+2)\,dx = \frac{125}{6} = 20.83$

h $A = \int_{-2}^{4.5} -x^2 + 2x + 15dx = 80.71$

i $y = 0$ is for $x = -3$ and $x = 5$.

$A = \int_{-3}^{5} -x^2 + 2x + 15dx = \frac{256}{3} = 85.33$

j $y = 0$ for $3 = e^x$ or $x = \ln 3$

$A = \int_{-1}^{\ln 3} 3 - e^x dx = 3.66$

k $y = 0$ is for $x = -\sqrt[3]{5} - 2$

$A = \int_{-2-5^{\frac{1}{3}}}^{0} (x+2)^3 + 5dx = 20.41$

2 **a** $x = -2$ or 2.

b The curve cuts the y axis at $(0,4)$.

c $A_1 = 5 \times 4 = 20.$

d $A_2 = \int_{0}^{2} -x^2 + 4dx = \frac{16}{3} = 5.33$

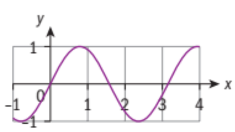
e $A = A_1 + A_2 = 25.33.$

3 Use your GDC to solve: $\int_{0}^{a} -(x+1)(x-5)\,dx = 24, a = 3.$

4 Use your GDC to solve: $\int_{-3}^{a} 2^{-x}dx = 9, a = -0.8169.$

Exercise 11C

1 **a**



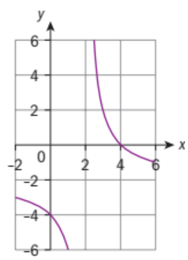
The x intercepts are at $2x = n\pi$ or $x = \frac{n\pi}{2}$ where $\frac{\pi}{2} \approx 1.57$.

b **i** $A = \int_{0}^{0.5} \sin 2x\,dx = 0.230$

ii $A = -\int_{2}^{3} \sin 2x\,dx = 0.807$

iii $A = \int_0^{\frac{\pi}{2}} \sin 2x\,dx - \int_{\frac{\pi}{2}}^{2.5} \sin 2x\,dx = 1.64.$

2 **a**



b The function is not defined at $x = 2$.

c The x intercept is at x = 4.

i $A_1 = \int_3^4 \frac{4}{x-2} - 2dx = 0.773$

ii $A_2 = -\int_4^6 \frac{4}{x-2} - 2dx = 1.227.$

d $A = A_1 + A_2 = 2.00.$

Exercise 11D

1 $A = \frac{1}{2} \times 2 \times (5 + 4 + 2(7 + 6 + 10)) = 55.$

2 $A = \frac{1}{2} \times 1.5 \times (1 + 0 + 2(4 + 2 + 5.5)) = 18.$

3 $A = \frac{1}{2} \times 0.5 \times (2 + 5.4 + 2(2.7 + 3 + 3 + 3.3 + 4)) = 9.85.$

4 $A = \frac{1}{2} \times 0.6 \times (5 + 3.25 + 2(4.32 + 3.87 + 3.57 + 3.37)) = 11.55.$

5 **a**

x	0	0.8	1.6	2.4	3.2	4.0
y	0	0.8944	1.2649	1.5492	1.7889	2.0000

$A = \frac{1}{2} \times 0.8 \times (0 + 2 + 2(0.8944 + 1.2649 + 1.5492 + 1.7889)) = 5.198$.

b

x	-1.00	0.25	1.50	2.75	4.00
y	-2.00	0.50	3.00	5.50	8.00

$A = \frac{1}{2} \times 1.25 \times (-2 + 8 + 2(0.5 + 3 + 5.5)) = 15.$

c

x	2.0	2.5	3.0	3.5	4.0	4.5	5.0
y	6.0000	5.0000	4.3333	3.8571	3.5000	3.2222	3.0000

$A = \frac{1}{2} \times 0.5 \times (6 + 3 + 2(5 + 4.3333 + 3.8571 + 3.5 + 3.2222)) = 12.21.$

d

x	0	1	2	3	4	5
y	0	4	9	12	10	0

$A = \frac{1}{2} \times 1 \times \left(0 + 0 + 2\left(4 + 9 + 12 + 10\right)\right) = 35.$

6 a

x	0	2	4	6	8
y	4	6	5.5	6.2	5

$A_1 = \frac{1}{2} \times 2 \times \left(4 + 5 + 2\left(6 + 5.5 + 6.2\right)\right) = 44.4\,\text{km}^2$

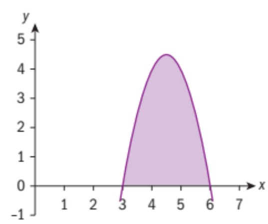
b

x	0	2	4	6	8
y	4	3	2.2	3.4	5

$A_2 = \frac{1}{2} \times 2 \times \left(4 + 5 + 2\left(3 + 2.2 + 3.4\right)\right) = 26.2\,\text{km}^2$

c $A = A_1 - A_2 = 18.2\,\text{km}^2$

7 a



b i $A = \int_3^6 -2\left(x-3\right)\left(x-6\right)dx,$ **ii** $A = 9.$

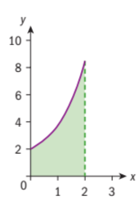
c

x	3	3.5	4	4.5	5	5.5	6
y	0	2.5	4	4.5	4	2.5	0

$A_2 = \frac{1}{2} \times \frac{1}{2} \times \left(0 + 0 + 2\left(2.5 + 4 + 4.5 + 4 + 2.5\right)\right) = 8.75$

d Percentage error $= \frac{0.25}{9} \times 100\% = 2.78\%.$

8 a



b i $A = \int_0^2 1 + e^x dx,$ **ii** $A = 8.389.$

c

x	0	0.4	0.8	1.2	1.6	2.0
y	2.0000	2.4918	3.2255	4.3201	5.9530	8.3891

$A_2 = \frac{1}{2} \times 0.4 \times \left(2.0000 + 8.3891 + 2\left(2.4918 + 3.2255 + 4.3201 + 5.9530\right)\right) = 8.474$

d Percentage error $= \frac{0.085}{8.389} \times 100\% = 1.01\%$.

9 Since the points cover half of the window, which is symmetrical, thens

$A = 2 \times \frac{1}{2} \times 1 \times \left(0 + 5.5 + 2\left(1.8 + 3.7 + 5\right)\right) = 26.5.$

10 a $A = 2 \times \frac{1}{2} \times 1 \times \left(6 + 10 + 2\left(8.5 + 9.4\right)\right) = 51.8$

b Find coefficients of $y = ax^3 + bx^2 + cx + d$ such that all given points are on the curve:

$a = \frac{13}{60}, b = -\frac{29}{20}, c = \frac{56}{15}, d = 6.$ Then, $A = 2\int_0^3 y\left(x\right) dx = 52.3.$

c Percentage error $= \frac{0.5}{52.3} \times 100\% = 0.96\%$.

Exercise 11E

1 a $10x + c$ **b** $0.2x^3 + c$ **c** $\frac{1}{6}x^6 + c$ **d** $7x - x^2 + c$

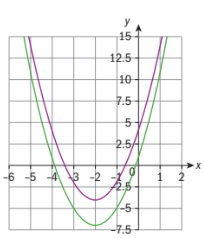
e $x + x^2 + c$ **f** $5x + \frac{1}{2}x^2 - \frac{1}{9}x^3 + c$ **g** $-\frac{1}{2}x^2 + \frac{1}{4}x^3 + 0.5x + c$

h $x - \frac{1}{2}x^2 + \frac{1}{8}x^4 + c$ **i** $\frac{1}{3}x^3 - \frac{1}{4}x^2 + 4x + c$

2 a $f'\left(x\right) = 2x - \frac{1}{3}$ **b** $\int f(x)dx = \frac{1}{3}x^3 - \frac{1}{6}x^2 + 4x + c$

3 a $\frac{1}{2}t^2 - t^3 + c$ **b** $t^4 - \frac{3}{2}t^2 + t + c$

4 a $f\left(x\right) = 2x^2 + 8x + c,$



i $c = 1$ **ii** $c = 4$

b i $f\left(-2\right) = -7$ **ii** $f\left(-2\right) = -4.$

5 a $-2\cos x + c$ **b** $4\sin x + c$ **c** $-\frac{1}{3}\cos x + c$ **d** $3\tan x + c$

6 **a** $2e^x + c$ **b** $3e^x + \ln|x| + c$ **c** $4\ln|x| - 3e^x + c$

7 **a** $\frac{2}{3}e^{3x} + c$ **b** $-\frac{2}{5}\cos 5t + c$ **c** $4x + \frac{2}{3}e^{-3x} + c$ **d** $\frac{1}{2}\sin 2t + \frac{1}{3}\cos 3t + c$

8 **a** $\frac{2\sqrt{5}}{3}t^{\frac{3}{2}} + c$ **b** $3\pi x^{\frac{1}{3}} + c$ **c** $\frac{1}{4}a^4 - a^3 + 2a + c$ **d** $2\sqrt{x} + c$

e $\frac{1}{2}x^2 + 2x + 3\ln|x| + c$ **f** $\frac{1}{2}e^{2x} - \frac{3}{4}\cos 4x + c$

g $\frac{2}{9}x^{\frac{3}{2}} - \frac{1}{12}\sin 4x + c$

9 **a** **i** $\frac{dy}{dx} = -\frac{\cos x}{\sin^2 x}$ **ii** $\frac{-1}{\sin x} + c$.

b **i** $\frac{dy}{dx} = \frac{2}{\sqrt{4x-1}}$ **ii** $\frac{1}{2}\sqrt{4x-1} + c$.

10 $y(x) = \frac{1}{2}x^2 + \frac{1}{15}x^3 + 2x + c,\ y(4) = \frac{304}{15} + c = 3,\ c = -\frac{259}{15}.$

Hence, $y = \frac{1}{2}x^2 + \frac{1}{15}x^3 + 2x - \frac{259}{15}$

11 $f(x) = 3x - \frac{1}{2}x^2 + c,\ f(3) = \frac{9}{2} + c = 2,\ c = -\frac{5}{2}.$

Hence, $f(x) = 3x - \frac{1}{2}x^2 - \frac{5}{2}$

12 $y(x) = \frac{1}{2}\sin 2x - \cos 2x + c,\ y\left(\frac{\pi}{4}\right) = \frac{1}{2} + c = 5,\ c = \frac{9}{2}.$

Hence, $y(x) = \frac{1}{2}\sin 2x - \cos 2x + \frac{9}{2}.$

13 $y(x) = 2e^{4x} + x^3 + x + c,\ y(0) = 2 + c = 4,\ c = 2.$

Hence, $y(x) = 2e^{4x} + x^3 + x + 2.$

Exercise 11F

1 $dx = \frac{1}{4}du.$

a $\int u^2 \frac{1}{4}du = \frac{1}{12}u^3 + c = \frac{1}{12}(4x-1)^3 + c$ **b** $\int \frac{1}{4u^2}du = -\frac{1}{4u} + c = -\frac{1}{4(4x-1)} + c$

c $\int \frac{1}{4\sqrt{u}}du = \frac{1}{2}\sqrt{u} + c = \frac{1}{2}\sqrt{4x-1} + c$ **d** $\int \frac{3}{4u}du = \frac{3}{4}\ln|u| + c = \frac{3}{4}\ln|4x-1| + c$

2 **a** $4x^2 + 12x + c$

b Let $u = 1 + x^2, du = 2x dx$, then $\int e^u du = e^u + c = e^{1+x^2} + c$.

c Let $u = 4 + 2x, du = 2dx$, then $\int 2\sec^2 u du = 2\tan u + c = 2\tan(4 + 2x) + c$.

d Let $u = 2x + 3, du = 2dx$, then $\int \frac{1}{2u} du = \frac{1}{2}\ln|u| + c = \frac{1}{2}\ln|2x + 3| + c$.

e Let $u = x^2 + 2, du = 2x dx$, then $\int 2\cos u du = 2\sin u + c = 2\sin(x^2 + 2) + c$.

f Let $u = \sin x, du = \cos x dx$, then $\int u^2 du = \frac{1}{3}u^3 + c = \frac{1}{3}\sin^3 x + c$.

3 Integrate to get $y = \frac{1}{2}x^2 + \ln|x + 2| + c$. Since $y(0) = 3$, then $3 = \ln 2 + c$, giving $c = 3 - \ln 2$.

Hence, $y = \frac{1}{2}x^2 + \ln|x + 2| + 3 - \ln 2$.

4 a Let $u = \sin 3x, du = 3\cos 3x dx$, then $\int \frac{1}{3} e^u du = \frac{1}{3}e^u + c = \frac{1}{3}e^{\sin 3x} + c$.

b Let $u = x^2 - 6x + 4, du = (2x - 6)dx$, then $\int \frac{1}{2}u^5 du = \frac{1}{12}u^6 + c = \frac{1}{12}(x^2 - 6x + 4)^6 + c$.

5 a $u = \cos x$ so $du = -\sin x\, dx$, giving $\int \frac{\sin x}{\cos x} dx = \int -\frac{du}{u} = -\ln|\cos x| + c$.

b $\frac{dy}{dx} = \tan x + c$, $3 = \tan\frac{\pi}{4} + c$, $c = 2$. Integrate again to get

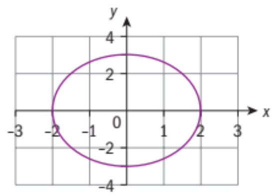
$y = -\ln|\cos x| + 2x + c$, $4 = -\ln|1| + c$, $c = 4$.

Hence, $y = -\ln|\cos x| + 2x + 4$.

6 a Let $u = 4 - x^2, du = -2x dx$, then $y = \int \frac{3}{4\sqrt{u}} du = \frac{3}{2}\sqrt{u} + c = \frac{3}{2}\sqrt{4 - x^2} + c$. Applying boundary

conditions gives $c = 0$, so $\frac{y^2}{9} + \frac{x^2}{4} = 1$.

b When $x = 0$, $y = \pm 3$.



c $A = 2\int_{-2}^{2} \frac{3}{2}\sqrt{4 - x^2} dx = 18.85$.

d $A = 6\pi$.

7 a Let $u = n^2 + 1, du = 2n dn$, then

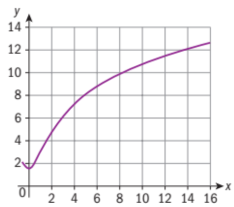
$C = \int \frac{2du}{u} = 2\ln(n^2 + 1) + c$.

$C(0) = c = 1.5,\ C(n) = 2\ln(n^2+1) + 1.5.$

b $C(7) = 2\ln 50 + 1.5 = 9.324.$

The cost of producing 7 items is €9 324.

c



As n increases, the cost grows more and more slowly because $\dfrac{dC}{dn} \to 0$ as $n \to \infty$.

8 a Let $u = 1 + 12e^{-0.5t}, du = -6e^{-0.5t}$, then

$R = \int -\frac{3}{5}\frac{du}{u^2} = \frac{3}{5u} + c = \frac{3}{5\left(1+12e^{-0.5t}\right)} + c,\ R(0) = \frac{3}{65} + c = \frac{1}{5},\ c = \frac{2}{13}.$

Hence, $R = \dfrac{3}{5\left(1+12e^{-0.5t}\right)} + \dfrac{2}{13}.$

b $R(t) = 0.7$ for $t = 9.6$ hours

c As $t \to \infty, R(t) \to \frac{3}{5} + \frac{2}{13} = \frac{49}{65} = 75\%.$

9 a The total rate of change of water in the tank is $C'(t) = S'(t) - R'(t) = \dfrac{10t}{1+2t^2} - 5\sin\left(\dfrac{t}{120}\right)$. A plot of this function shows that at $t = 0$, $C'(t) = 0$ and for $0 < t \le 5$, then $C'(t) > 0$. That is, the amount of water in the tank is increasing for $0 < t \le 5$.

b $T(t) = 25 + \int_0^t C'(t)\,dt = \dfrac{5}{2}\left(\ln\left(2t^2+1\right) + 240\cos\left(\dfrac{t}{120}\right)\right) - 575.$

c Since the amount of water is always increasing, the maximum is obtained at $t = 5$ hours, $T(5) = 34\,\text{m}^3$.

Exercise 11G

1 a $A = \int_0^1 e^{2x} + 1dx = \left[\frac{1}{2}e^{2x} + x\right]_0^1 = \frac{1}{2}\left(1+e^2\right)$

b $A = \int_2^4 \frac{2}{x}dx = \left[2\ln|x|\right]_2^4 = 2\ln 2$

c $u = x^2 - 5$. When $x = 3$ then $u = 4$. When $x = 4$, then $u = 11$.

$$A = \int_4^{11} \frac{1}{2}\sqrt{u}\,du = \frac{1}{2}\left[\frac{2}{3}u^{\frac{2}{3}}\right]_4^{11} = \frac{1}{3}\left[11\sqrt{11} - 8\right]$$

2 $\int_1^4 \frac{x}{x^2+2}dx = \int_3^{18} \frac{du}{2u} = \left[\frac{1}{2}\ln|u|\right]_3^{18} = \frac{1}{2}\ln 6$, where $u = x^2+2$.

3 $\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \cos 2x dx = \left[\frac{1}{2}\sin 2x\right]_{\frac{\pi}{6}}^{\frac{\pi}{4}} = \frac{1}{2}\left(1-\frac{\sqrt{3}}{2}\right)$.

4 $u = t^2+2$

Energy $E = \int_0^{t_1} \frac{4t}{t^2+2}dt = 2\int_2^{t_1^2+2} \frac{du}{u} = 2\ln\frac{t_1^2+2}{2}$.

5 Area $A = \frac{1}{2}\pi\times 4^2 + \int_{-2}^{2} x^4 - 5x^2 + 4dx = 8\pi + \left[\frac{1}{5}x^5 - \frac{5}{3}x^3 + 4x\right]_{-2}^{2} = 8\pi + \frac{32}{15} = 27.3$ cm²

Exercise 11H

1 a $A = \int_{-0.5}^{1} -2x^2 + x - 1dx = -1.875$.

b $A = \int_0^3 |\,(x-3)^3\,|\,dx = \frac{81}{4} = 20.25$.

c $A = \int_{0.5}^{e} |\ln x|dx = 1.15$.

d $A = \int_0^4 \left|(x-4)-(x^2-3x-4)\right|dx = \frac{32}{3}$

e Curves intersect when $x^2 = 4-x$ or $x^2+x-4=0$ giving $x = -\frac{1}{2}(1\pm\sqrt{17})$. Using the positive root, $A = \int_0^{\frac{1}{2}(\sqrt{17}-1)} x^2 dx + \int_{\frac{1}{2}(\sqrt{17}-1)}^{4} 4-x dx = 4.24$

f $A = \int_0^2 \left|e^x - e^{-x}\right|dx = 5.52$.

2 $x = \frac{3}{y}$

$A = \int_1^3 \frac{3}{y}dy = 3.30$.

3 $A = \int_0^{2\pi} |\sin y|dy = 4$.

4 a $y=0$ for $bx = n\pi$. For $n=1$ then $x=5$ so $5b=\pi$ giving $b=\frac{\pi}{5}$.

For $x=2.5$, $4 = a\sin\left(\frac{\pi}{5}\times 2.5\right) = a\sin\left(\frac{\pi}{2}\right) = a$.

b Equation for the inner surface of the roof is the equation for the outside of the roof translated in y direction: $y = 4\sin\frac{\pi}{5}x - 1$.

c For the inner surface, $y=0$ when $\sin\frac{\pi x}{5}=\frac{1}{4}$, that is $x=\frac{5}{\pi}\sin^{-1}\left(\frac{1}{4}\right)=0.40$ and $x=\frac{5}{\pi}\left[\pi-\sin^{-1}\left(\frac{1}{4}\right)\right]=4.60$.

Area of the cross-section: $A=\int_0^5 4\sin\frac{\pi}{5}x\,dx-\int_{0.40}^{4.60}4\sin\frac{\pi}{5}x-1\,dx=4.60\ \text{m}^2$.

Hence, the volume $V=15A=69.0\ \text{m}^3$.

5 a i $p_a=\frac{1}{2}\ln(a+1)$

$$p=\frac{1}{2}\ln(q+1)$$
$$e^{2p}=q+1$$
$$q=e^{2p}-1$$
$$A_1=\int_0^{p_a}(e^{2p}-1)dp$$
$$=\left[\frac{e^{2p}}{2}-p\right]_0^{p_a}$$
$$=\left(\frac{a+1}{2}-\frac{1}{2}\ln(a+1)\right)-\frac{1}{2}$$
$$=\frac{a}{2}-\frac{1}{2}\ln(a+1)$$

ii Since $A_1+A_2=p_a a$ then

$$A_2=\frac{a}{2}\ln(a+1)-\frac{a}{2}+\frac{1}{2}\ln(a+1)$$
$$=\frac{1}{2}(a+1)\ln(a+1)-\frac{a}{2}$$

b Solve $A_1\ge 0.5\,A_2$, *i.e.* $\frac{a}{2}-\frac{1}{2}\ln(a+1)\ge 0.5\left(\frac{a+1}{2}\ln(a+1)-\frac{a}{2}\right)$.

$a\le 7.577$ so maximum integer value of a is 7.

Exercise 11I

1 a $V=\int_0^1 \pi a^2(x)\,dx=\frac{\pi}{3}$

b $V=\int_0^1 \pi b^2(x)\,dx=\frac{\pi}{5}$

c $V=\int_0^1 \pi c^2(x)\,dx=\frac{\pi}{7}$

d $V=\int_0^1 \pi e^2(x)\,dx=\frac{1}{2}(e^2-1)\pi$

2 **a** $a(y) = y, 0 \le y \le 1, V = \int_0^1 \pi a^2(y)\,dy = \frac{\pi}{3}$ **b** $b(y) = \sqrt{y}, 0 \le y \le 1, V = \int_0^1 \pi b^2(y)\,dy = \frac{\pi}{2}$

c $c(y) = y^{\frac{1}{3}}, 0 \le y \le 1, V = \int_0^1 \pi c^2(y)\,dy = \frac{3\pi}{5}$ **d** $e(y) = \ln y, 1 \le y \le e, V = \int_1^e \pi e^2(x)\,dx = (e-2)\pi.$

3 **a** $F(x) = \int\left(\frac{x}{100}\sqrt{900 - x^2}\right)^2 dx = \frac{300}{10000}x^3 - \frac{x^5}{50000} + c$

b $V = \pi\left(F(30) - F(0)\right) = 324\pi = 1018\,\text{mm}^3.$

c For one earring, the cost = $30 \times 19 \times 1.018 = £580.$

4 **a** Equation of the line $y = 4 - \frac{4}{3}x, \text{or } x = 3 - \frac{3}{4}y$, hence,

i $V = \int_0^3 \pi\left(4 - \frac{4}{3}x\right)^2 dx$ **ii** $V = \int_0^4 \pi\left(3 - \frac{3}{4}y\right)^2 dy.$

b **i** $V = 16\pi$ use the formula with $r = 4, h = 3$ to obtain the same answer.

ii $V = 12\pi$ use the formula with $r = 3, h = 4$ to obtain the same answer.

5 **a** From the figure, model $s(x)$ corresponds to the shape of the football better. Hence,

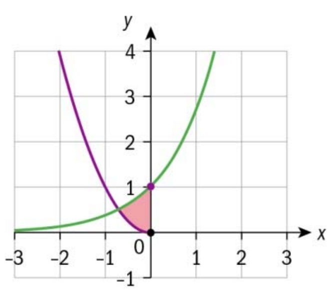
$V = \int_{-7.5}^{7.5} \pi\left(-4 + \sqrt{72.25 - x^2}\right)^2 dx = 558\,\text{units}^3.$

b **i** $V = \left(\frac{11}{15}\right)^3 \times 558 = 220\,\text{in}^3$

ii 1 in^3 = 16.39 cm^3

Hence, 220 in^3 = 220 × 16.39 = 3606 cm^3

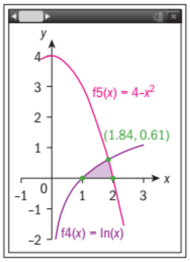
6 **a** **i**



ii If $x^2 = e^x$ then $x = -0.70$. $V = \int_{-0.70}^{0} \pi\left(e^{2x} - x^4\right)dx$

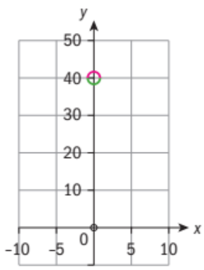
iii 1.08

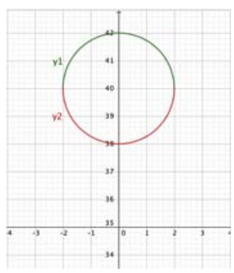
b **i**



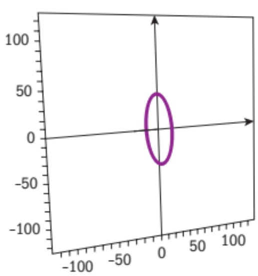
ii $V = \int_{1}^{1.84} \pi \ln^2 x dx + \int_{1.84}^{2} \pi \left(4 - x^2\right)^2 dx$ **iii** 0.44

7 a





b The shape is a torus



c $V = \int_{-2}^{2} \pi \left(\left(40 + \sqrt{4 - x^2}\right)^2 - \left(40 - \sqrt{4 - x^2}\right)^2 \right) dx = 3158$ cubic units

8 a Consider triangles CAD and DAB. $AC = AB = r, CD = DB$ and AD is common. Hence the triangles are congruent. Hence, $\angle ADB = \angle ADC = 90°$ as the two angles are equal and add up to 180°

b We have $CB = 365$ m and $AD = r - 50$, so using Pythagoras theorem:

$$(r - 50)^2 + \left(\frac{1}{2} \times 365\right)^2 = r^2$$

$r = 358$ m.

c The volume is formed by rotating the section of the curve $x^2 + y^2 = 358^2$ about the *y* axis between $y = 358 - 50 = 308$ and $y = 358$

$$\pi \int_{308}^{358} \left(x^2\right) dy = \pi \int_{308}^{358} \left(358^2 - y^2\right) dy = 2.7 \times 10^6$$

Exercise 11J

1 **a** $v(t) = \int \frac{3}{(t+1)^2}\,dt = -\frac{3}{t+1} + c$. Since $v(0) = 0$, $c = 3$. Hence, $v(t) = -\frac{3}{t+1} + 3$.

$d(t) = \int v(t)\,dt = \int -\frac{3}{t+1} + 3dt = 3(t - \ln(1+t)) + c$. Since $d(0) = 0$, $c = 0$. Hence,
$d(t) = 3(t - \ln(1+t))$

b Solve $50 = 3(t - \ln(1+t)), t = 20$.

2 **a** **i** Acceleration $a(t) = \frac{dv}{dt} = 3e^{2\sin 3t}\left(2\cos^2 3t - \sin 3t\right)$

ii $a(t) = -0.5$ at $t = 0.3\,\text{s}$

b Displacement $s(t) = \int_0^t v(r)\,dr = \frac{1}{6}\left(e^{2\sin 3t} - 1\right)$.

c Maximum value of displacement occurs when the exponent is the largest, $s_{\max} = \frac{1}{6}\left(e^2 - 1\right)$.

3 **a** Acceleration $a(t) = \frac{dv}{dt} = e^{-kt^2}\left(1 - 2kt^2\right)$, maximum acceleration occurs at $t = 0, a_{\max} = 1\,\text{cm/s}^2$

b Displacement $d(t) = -\frac{1}{2k}e^{-kt^2} + c$. Since $d(0) = 1\,\text{cm}$, $1 = -\frac{1}{2k} + c$ and $c = 1 + \frac{1}{2k}$. Hence,
$d(t) = \frac{1}{2k}\left(2k + 1 - e^{-kt^2}\right)$.

Maximum displacement of the particle occurs when $t \to \infty$ so $d_{\max} = 1 + \frac{1}{2k} \geq 5$ giving $k \leq \frac{1}{8}$.

Hence, $0 < k \leq \frac{1}{8}$.

4 **a** Velocity of the projectile $v(t) = \int g\,dt = gt + v_0$

Height of the projectile $s(t) = \int v(t)\,dt = \frac{1}{2}gt^2 + v_0 t + y_0$.

b Maximum height occurs when $v(t) = 0$, $t_{\max} = -\frac{v_0}{g}$, $s(t_{\max}) = s_{max} = y_0 - \frac{v_0^2}{2g}$.

c For $v_0 = 310\,\text{ms}^{-1}$, $y_0 = 10\,\text{m}$, $g = -9.8$ then $s_{max} = 4913\,\text{m} > 4\,\text{km}$, so the rocket will reach the target height.

Exercise 11K

1 **a** Net change: $T(8) = \int_0^8 T'(t)\,dt = -21.3^o C$.

b Total change: $\Delta T(8) = \int_0^8 |T'(t)|\,dt = 76.8^o C$.

2 **a** **i** Total distance $= \int_0^5 |10|\, dt = 50\,\text{m}$, displacement $= \int_0^5 10\,dt = 50\,\text{m}$.

ii Total distance $= \int_0^5 |10-2t|\, dt = 25\,\text{m}$, displacement $= \int_0^5 10-2t\,dt = 25\,\text{m}$.

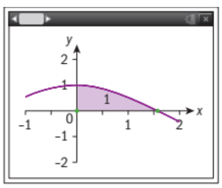
iii Total distance $= \int_0^5 |10-4t|\, dt = 25\,\text{m}$, displacement $= \int_0^5 10-4t\,dt = 0\,\text{m}$.

iv Total distance $= \int_0^{2.5} |10-4t|\, dt + \int_{2.5}^5 |4t-10|\, dt = 25\,\text{m}$, displacement

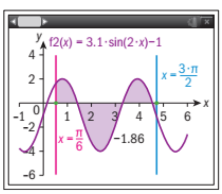
$$= \int_0^{2.5} 10-4t\,dt + \int_{2.5}^5 4t-10\,dt = 25\,\text{m}.$$

b Only particle C changes direction since its velocity goes from positive to negative.

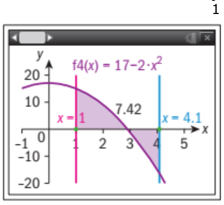
3 **a** Total distance $= \int_0^{\frac{\pi}{2}} |\cos t|\, dt = 1\,\text{km}$, displacement $= \int_0^{\frac{\pi}{2}} \cos t\,dt = 1\,\text{km}$.



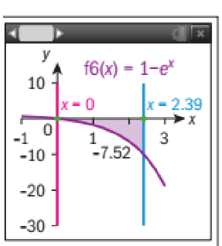
b Total distance $= \int_{\frac{\pi}{6}}^{\frac{3\pi}{2}} |3.1\sin 2t - 1|\, dt = 7.97\,\text{km}$, displacement $= \int_{\frac{\pi}{6}}^{\frac{3\pi}{2}} 3.1\sin 2t - 1\,dt = -1.86\,\text{km}$.



c Total distance $= \int_1^{4.1} |17-2t^2|\, dt = 26.0\,\text{km}$, displacement $= \int_1^{4.1} 17-2t^2\,dt = 7.42\,\text{km}$.



d Total distance $= \int_0^{2.39} |1-e^t|\, dt = 7.52\,\text{km}$, displacement $= \int_0^{2.39} 1-e^t\,dt = -7.52\,\text{km}$.

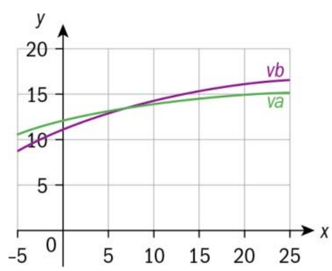


4 **a** **i** $d = 20\,\text{m}$ **ii** $d = 20 + \int_5^{10} \left| 4 - \frac{4}{25}(t-5)^2 \right| dt = 33.3\,\text{m}$

b For $t > 10$ $d(t) = \int v\,dt = -0.8t^2 + 16t + c$. Since $d(10) = 33.3$, $c = -46.7$. Thus, $s(t) = -0.8t^2 + 16t - 46.7$ and $s(t) = 0$ for $t = 16.5$ s (since $t > 10$).

5 **a** Displacement $\int_0^5 v(t)\,dt = -0.0154\,\text{cm}$ **b** Total distance $\int_0^5 |v(t)|\,dt = 6.53\,\text{cm}$.

6 **a** **i**



ii $v_a = v_b, t = 16 \log \frac{3}{2} = 6.49\,\text{s}$

b Distance above the ground $d = 1000 - \int_0^{6.49} v_a(t)\,dt = 917.5\,\text{m}$.

7 $E(\pi) - E(0) = \int_0^{\pi} \sin 0.5t + a\,dt = 2 + \pi a = 4, a = \frac{2}{\pi}$ mJs^{-1}.

8 $A(4) - A(2) = \int_2^4 2e^{rt}\,dt = \frac{4}{r} e^{3r} \sinh r = 7.2\,\text{cm}^2, r = 0.194\,\text{s}^{-1}$.

9 **a** The population of mosquitoes is a minimum when

$$\frac{dN}{dt} = 0, \quad t \neq 0,$$

$$\cos\left(\frac{\pi t}{6} - \frac{1}{2}\right) = 0$$

$$\frac{\pi t}{6} - \frac{1}{2} = \frac{\pi}{2} + n\pi$$

For $n = 1$, $t = 3 + \frac{3}{\pi} \approx 3.95$.

i.e. in April

b **i** $\int_0^{12} \frac{dN}{dt}\,dt = -13.6$. Hence, the population decreases by 136 000.

ii $\int_{12}^{24} \frac{dN}{dt}\,dt = -5.73$. Hence, the population decreases by 57 300.

c $-\frac{19.33}{2} = -9.67$. Hence, the average annual decrease in population is 96 700.

Exercise 11L

1 This is a separable equation: $\int \frac{4}{y} dy = \int x dx$, $4\ln y = \frac{1}{2}x^2 + c$, $y = e^{\frac{1}{8}x^2} e^{\frac{1}{4}c}$. Apply the boundary condition to get $y = 2e^{\frac{1}{8}x^2}$.

2 This is a separable equation: $\int \frac{1}{y} dy = \int \frac{2x}{x^2 - 1} dx, \ln y = \ln\left|x^2 - 1\right| + c, y = y_0\left|x^2 - 1\right|$. Apply the boundary condition to get $y = 3\left|x^2 - 1\right|$.

3 **a** $\frac{dM}{dt} = -kM$. This is a separable equation: $\int \frac{1}{M} dM = \int -kdt, \ln M = -kt + c, M = Ae^{-kt}$.

b $A = 40$ is the initial amount. Hence, $20 = 40e^{-2k}, k = \frac{1}{2}\ln 2$.

4 **a** $\frac{dy}{dx} = -\frac{x}{\sqrt{4 - x^2}}$

b $\frac{dy}{dx} = -\frac{x}{\sqrt{4 - x^2}} = -\frac{x}{y}$

c This is a separable equation: $\int y dy = -\int x dx$, $\frac{1}{2}y^2 = -\frac{1}{2}x^2 + \frac{c}{2}$, $y = \pm\sqrt{c - x^2}$.

5 **a** This is a separable equation: $\int e^{-y} dy = \int e^x dx, -e^{-y} = e^x + c$, $y = -\ln\left(-c - e^x\right)$.

b This is a separable equation: $\int y + \frac{1}{y} dy = \int \left(1 + \cos x\right) dx$, $\frac{1}{2}y^2 + \ln y = x + \sin x + c$.

6 **a** This is a separable equation:

$$\sqrt{2}\int \sqrt{y} dy = \int \sqrt{x} dx$$

$$\sqrt{2}\frac{2}{3}y^{\frac{3}{2}} = \frac{2}{3}x^{\frac{3}{2}} + \frac{2}{3}c$$

$$y = \left(\frac{x^{\frac{3}{2}} + c}{\sqrt{2}}\right)^{\frac{2}{3}} = \frac{1}{\sqrt[3]{2}}(x^{\frac{3}{2}} + c)^{\frac{2}{3}}$$

Apply the boundary condition to get:

$$2 = \frac{1}{\sqrt[3]{2}}(1 + c)^{\frac{2}{3}}$$

$$1 + c = 4$$

$$c = 3$$

So, $y = \frac{1}{\sqrt[3]{2}}(x^{\frac{3}{2}} + 3)^{\frac{2}{3}}$

b This is a separable equation: $\int y dy = \int 2x + \sin x dx, \frac{1}{2}y^2 = x^2 - \cos x + c$. Apply the boundary condition to get $\frac{1}{2}y^2 = x^2 - \cos x + \frac{3}{2}$.

c This is a separable equation: $\int \cot y dy = \int \cot x dx$, $\ln \sin y = \ln \sin x + c$. Apply the boundary condition to get $\sin y = \sin x$, $y = x + 2\pi n$, $n \in$.

7 **a** At $t = 0$ because the temperature difference between the coffee and the room is the largest.

b $\frac{dT}{dt} = -k(T-24)$, $T(0) = 75°C$.

c This is a separable equation: $\int \frac{1}{T-24}dT = \int -kdt$, $\ln(T-24) = -kt + c$, $T = 51e^{-kt} + 24$.

d $e^{-kt} \to 0$ as $t \to \infty$, so this term is never negative.

8 a Differential equation governing the number of bacteria N is $\frac{dN}{dt} = kN$. This is a separable equation: $\int \frac{1}{N}dN = \int kdt$, $\ln N = kt + c$, $N(t) = 10^4 e^{kt}$. To find k, $3\times10^5 = 10^4 e^{3k}$, $k = \frac{1}{3}\ln 30$.

Since $e^{\frac{t}{3}\ln 30} = 30^{\frac{t}{3}}$, then $N(t) = 10^4 \times 30^{\frac{1}{3}t}$.

Hence, after 24 hours of growth there will be $N(24) = 10^4 \times 30^8 = 6.56\times10^{15}$.

b Since we can write $N = N_0 30^{\frac{t}{3}}$, then $2 = 30^{\frac{1}{3}t}$ and $t = \frac{\ln 8}{\ln 30} = 0.611$ hours.

9 a $\frac{dN}{dt} = -kN$ is a separable equation: $\int \frac{1}{N}dN = \int -kdt$, $\ln N = -kt + c$. If $N(0) = N_0$ then $N(t) = N_0 e^{-kt}$. For half-life T, $\frac{1}{2} = e^{-kT}$ giving $k = \frac{\ln 2}{T}$.

Hence, $N = N_0 2^{-\frac{t}{T}}$.

b For indium-111, we have $N(t) = 5\times 2^{-\frac{t}{2.8}}$. For $t = 1$ day, $N = 5\times 2^{-\frac{1}{2.8}} = 3.90$ g.

10 a The amount of water leaving the tank every second is $Av = A\sqrt{2gh}$. Hence, the rate of change of the total volume of water in the tank is $\frac{dV}{dt} = -A\sqrt{2gh}$.

b i $V = \pi r^2 h$, $\frac{dV}{dt} = -A\sqrt{\frac{2gV}{\pi r^2}} = -k\sqrt{V}$ **ii** $k = A\sqrt{\frac{2g}{\pi r^2}}$.

c Solve the separable differential equation: $\int \frac{dV}{\sqrt{V}} = -\int kdt$ giving $2\sqrt{V} = -kt + c$. Since $V(0) = V_0$, $2\sqrt{V} = 2\sqrt{V_0} - kt$. Hence, when $V = 0$ then $t = \frac{2\sqrt{V_0}}{k}$.

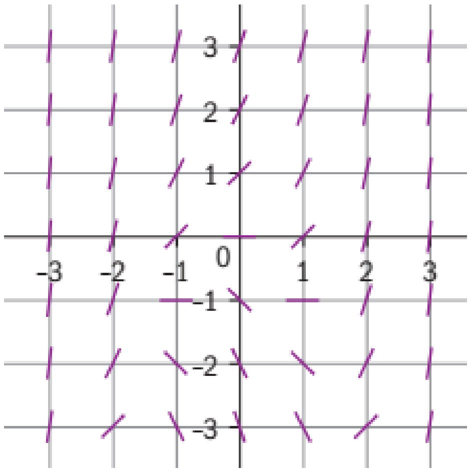
Exercise 11M

1 a

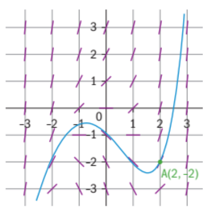
		x		
		1	2	3
y	1	2	5	10
	2	3	6	11

	3	4	7	12

b



c



2 **a** At (0,0), $\frac{dy}{dx} = -2$, which matches slope field III

b At (1,−1) , $\frac{dy}{dx} = 0$, which matches slope field I

c At (0,0) , $\frac{dy}{dx} = 2$, which matches slope field IV

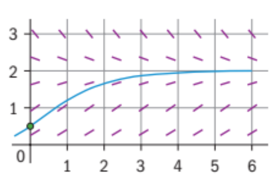
d At (2,2) , $\frac{dy}{dx} = 8$, which matches slope field II

3 **a** **i** $y' = 0$ for $y = \pm\frac{\pi}{2} - x$ **ii** $y' = 0$ for $y = \pm\frac{\pi}{2} + x$

b **i** B (gradient is zero on the line $y = \pm\frac{\pi}{2} - x$)

ii A (gradient is zero on the line $y = \pm\frac{\pi}{2} + x$)

4 **a,b**



c From the model, the maximum is for $\ln\left(\frac{T}{2}\right) = 0$, that is $T = 2$ cm^3 (if the tumour is bigger, it would shrink). The slope field confirms this.

d That a tumour has a maximum size is consistent with that it grows in an environment where the availability of nutrients is limited.

Exercise 11N

1

x	y
4.1	4.05
4.2	4.10
4.3	4.15
4.4	4.20
4.5	4.25

2

x	y
0.05	-0.90
0.10	-0.81
0.15	-0.71

3

x	y
1.025	0.075
1.050	0.145
1.075	0.211
1.100	0.274

4 **a**

x	y
1.1	0.90
1.2	0.801
1.3	0.70488
1.4	0.6132

1.5	0.52739

b This is a separable equation: $\int \frac{1}{y}dy = -\int x dx$, $\ln y = -\frac{1}{2}x^2 + c$, $y = e^{-\frac{1}{2}x^2+c}$. Apply the boundary condition to get $y = e^{-\frac{1}{2}x^2+\frac{1}{2}}$. Then, $y(1.5) = 0.53526...$, so the error is 1.5%.

5 a

x	y
1.1	3.30
1.2	3.63
1.3	3.99

b This is a separable equation: $\int \frac{1}{y}dy = \int \frac{2x}{1+x^2}dx$, $\ln y = \ln(1+x^2) + c$. Apply the boundary condition to get $y = \frac{3}{2}(1+x^2)$. Then, $y(1.3) = 4.04$, so the error is 1.2%.

6 a For $N = a + bt$,

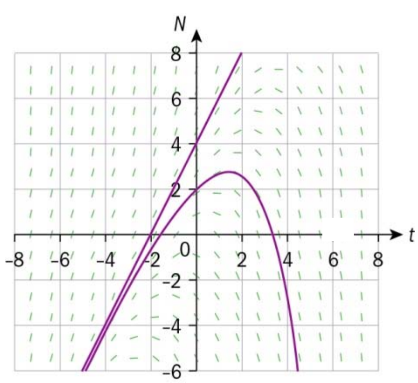
$$\frac{dN}{dt} = b$$

As $\frac{dN}{dt} = 0.5N - t$.

$$b = 0.5N - t \Rightarrow 2b = N - 2t \Rightarrow N = 2b + 2t$$

$b = 2$ and $a = 2b = 4$

b



c From the slope field, it is clear that any $N_0 < 4$ will give a curve that peaks and then declines, while $N_0 = 4$ gives growth at a constant rate and $N_0 > 4$ gives a population with an increasing rate of growth. Thus, the colony will become extinct if $N_0 < 4$ so the minimum is $N_0 = 4$, that is 40 meerkats.

d Maximum is reached when $\frac{dN}{dt} = 0 = 0.5N - 3$ giving $N = 6$. Thus, there are 60 meerkats when the population begins to decline.

e At $t = 3, N = 4 + 2t$ gives $N = 10$, so $\Delta N = 4$. That is, 40 more meerkats are needed to reach constant growth instead of decline.

f

t **(years)**	***N*** **(10 meerkats)**
3.1	12.3
3.2	12.6
3.3	12.9
3.4	13.2
3.5	13.6
3.6	13.9
3.7	14.2
3.8	14.6
3.9	14.9
4.0	15.3

After 1 year, the number of meerkats will be 153.

Chapter Review

1 **a** For $t \in (0, 30)$, the drone moves in the positive direction with constant velocity $v = 40$ km/h.

For $t \in (30, 60)$, the drone moves in the positive direction with decreasing speed, i.e. with acceleration $a = -\frac{4}{3}$ km/h^2.

For $t \in (60, 75)$, the drone moves in the negative direction with increasing speed, i.e. with acceleration $a = -\frac{4}{3}$ km/h^2.

For $t \in (75, 90)$, the drone moves in the negative direction with constant velocity $v = -20$ km/h.

b $t \in (30, 75)$

c $t \in (30, 60)$

d Distance $d = \int_0^{90} |v|\, dt$ which is equal to the total area between t axis and the curve:

$$d = (40 \times 30) + \left(\frac{1}{2} \times 30 \times 40\right) + \left(\frac{1}{2} \times 15 \times 20\right) + (15 \times 20) = 2250\,\text{km}.$$

e Final position of the drone:

$$p = \int_0^{90} v\,dt = (40\times30) + \left(\frac{1}{2}\times30\times40\right) - \left(\frac{1}{2}\times15\times20\right) - (15\times20) = 1350\,\text{km}.$$

2 **a** $d = \int_0^4 10-(x-2)^2\,dx + (2\times6) = 46.7\,\text{m}.$ **b** $t = 10\,\text{s}$

c Total distance travelled away is $\int_0^4 10-(x-2)^2\,dx + (2\times6) + \left(\frac{1}{2}\times4\times6\right) = 58.7\,\text{m}$. Distance travelled back in terms of the return time t:

$$d = \left(\frac{1}{2}\times2\times3\right) + \left[(t-12)\times3\right] = 3t - 33 = 58.7$$

$t = 30.6\,\text{s}$ for the return journey.

3 $d = \frac{1}{2}\times5\times\left(6.7+35.5+2\left(15.3+26+27.3+29.4+32.1\right)\right) = 756\,\text{m}$

4 Can be found analytically: **a,b,c,e,f,g,h**, need technology: **d,i**.

5 **a** $\frac{dP}{dt} = -aP$ for positive constant a.

b $\int\frac{dP}{P} = \int -a\,dt$ so $P(t) = P_0e^{-at}$. Apply boundary conditions

$P(0) = P_0 = 600\,000$ and $P(4) = 500\,000$ to obtain $a = \frac{1}{4}\ln\left(\frac{6}{5}\right)$. Thus,

$$P = 600\,000e^{-\frac{1}{4}\ln\left(\frac{6}{5}\right)t} = 600\,000e^{-0.04558t}\left(= 600\,000\left(\frac{6}{5}\right)^{-\frac{1}{4}t}.\right)$$

c Solve $0.4 = e^{-\frac{1}{4}\ln\left(\frac{6}{5}\right)t}$ to give $t = 20.1\,\text{s}$.

6 **a** Hemisphere **b** $V = \int_0^r \pi\left(r^2-x^2\right)dx$

c Volume of the sphere is $V = 2\int_0^r \pi\left(r^2-x^2\right)dx = 2\pi\left(r^3-\frac{1}{3}r^3\right) = \frac{4}{3}\pi r^3.$

7 **a**

x	y
0.1	3.60
0.2	4.39
0.3	5.45

b This is a separable equation:

$$\int\frac{1}{y}\,dy = \int 2(1+x)\,dx$$

$$\ln y = 2x + x^2 + c$$

After applying the boundary condition $y = 3e^{2x+x^2}$. Hence,
$y(0.1) = 3.70$, $y(0.2) = 4.66$, $y(0.3) = 5.98$ with absolute errors of 0.10, 0.27 and 0.53, respectively.

8 **a** $\frac{dV}{dt} = -k\sqrt{h}$, where k is a positive constant.

b

$$V = \pi 0.4^2 h$$
$$\frac{dV}{dt} = \frac{dV}{dh}\frac{dh}{dt} = \frac{4\pi}{25}\frac{dh}{dt}$$
$$\frac{4\pi}{25}\frac{dh}{dt} = -k\sqrt{h}.$$

c Solve the equation:

$$\int \frac{dh}{\sqrt{h}} = -\frac{25k}{4\pi}\int dt$$
$$2\sqrt{h} = -\frac{25k}{4\pi}t + c$$

Since $h = 1$ m when $t = 0$, $c = 2$. Thus,

$$h = \left(1 - \frac{25k}{8\pi}t\right)^2$$

Use $h(10) = 0.64$ and let $a = \frac{25k}{8\pi}$ to give

$$h = (1 - at)^2$$
$$0.64 = (1 - 10a)^2$$
$$a = \frac{1}{10}\left(1 - \sqrt{0.64}\right) = 0.02$$
$$h = (1 - 0.02t)^2$$

The tank will be empty when $h(t) = 0, t = 50$ min.

9 **a** Area $A = \int_{-2}^{2}\left|\left(2^x + 2^{-x}\right) - \left(\frac{17}{16}x^2\right)\right| dx = 5.15\,\text{m}^2$.

b The volume of the art installation $V = 40^3 \times 0.5 \times 5.15 = 164912\,\text{m}^3$.

Exam style questions

10 a 0.347

b Using $v = x^2 + 4$ then $dv = 2x\ dx$ so $\int\left(\frac{dv}{2v}\right) = \frac{1}{2}\ln v + c = \frac{1}{2}\ln\left(x^2 + 4\right) + c$

c $\int_0^2 \frac{x}{\left(x^2 + 4\right)} dx = \frac{1}{2}(\ln 8 - \ln 4) = \frac{\ln 2}{2}$.

11 a $\frac{1}{3}x^3+\frac{3}{2}x^2+x+c$

b $-\cos x+\frac{1}{2}\sin 2x+c$

c $e^x-e^{-x}+c$

d

$u=3x+1$

$du=3\,dx$

$\int\frac{u^6}{3}du=\frac{1}{21}(3x+1)^7+c$

e $\int\frac{x^2+x}{x}dx=\int x+1\,dx=\frac{1}{2}x^2+x+c$

12 $A=\int_0^{10}\left|\left(x^2-8x+23\right)-\left(x+9\right)\right|dx=65,$ so the area is $65\times10^4\ \text{m}^2$.

13 a $A=\frac{1}{2}\times1\times\left(0+0+2\begin{pmatrix}6.71+10.4+12.15+12.8+12.95+12.96+12.95\\+12.8+12.15+10.4+6.71\end{pmatrix}\right)$

$=122.98\,\text{m}^2$.

b $A_{true}=\int_0^{12}d\left(x\right)dx=124.42\,\text{m}^2$

c Percentage error $=\left|\frac{124.42-122.98}{124.42}\right|\times100\%=1.2\%$.

d When approximating this area using the trapezium rule, a small area above the trapeziums and below the actual curve is not taken into account. Hence, the estimate is smaller than the true value.

14 $V=\int_0^1\pi e^{2x^2}dx=7.43$.

15 a $v\left(t\right)=\int 4e^{-0.2t}dt=-20e^{-0.2t}+c$. Apply the initial condition to get $v\left(t\right)=20\left(1-e^{-0.2t}\right)$.

b $s\left(t\right)=\int 20\left(1-e^{-0.2t}\right)dt=20t+100e^{-0.2t}+c$. Apply the initial condition to get

$s\left(t\right)=20t+100\left(e^{-0.2t}-1\right)$.

c $v\left(t\right)\to20$ m/s as $t\to\infty$.

d $s\left(60\right)=1100\,\text{m}$.

16 a

x	y
1.25	2.50
1.50	3.28
1.75	4.51
2.00	6.49

b This is a separable equation: $\int \frac{1}{y}dy = \int x dx, \ln y = \frac{1}{2}x^2 + c, y = e^{\frac{1}{2}x^2}e^c$. Apply the boundary condition to get $e^c = 2e^{-\frac{1}{2}}$ so $y = 2e^{-\frac{1}{2}}e^{\frac{1}{2}x^2} = 1.21e^{\frac{1}{2}x^2}$.

c $y(2) = 2e^{\frac{3}{2}}$

d Absolute percentage error $= \left|\frac{6.49 - 8.96}{8.96}\right| \times 100\% = 28\%$.

17 a $2x \sin x + x^2 \cos x$

b $\int 4\cos x + 3(2x \sin x + x^2 \cos x)dx = 4\sin x + 3x^2 \sin x + c.$

18 a **i** isocline with $k = 4$

ii not an isocline, $\frac{dy}{dx}$ changes with x

iii isocline with $k = 1$

iv not an isocline because $\frac{dy}{dx} = \frac{2x^2 + 2x + 1}{2x^2 + 2x} \neq const.$

v isocline with $k = -1$.

b Turning point occurs when $\frac{dy}{dx} = 0$. This is impossible for $x \neq 0, y \neq 0$.

12 Modelling motion and change in two and three dimensions

Skills check

1 **a** $\int 6e^{3x} - \frac{1}{e^{2x}}dx = 2e^{3x} + \frac{1}{2e^{2x}} + c$

b $\int 3\cos(6x) - 4\sin(2x)dx = \frac{1}{2}\sin(6x) + 2\cos(2x) + c$

2 **a** 87.2°

b $\begin{pmatrix}2\\3\\2\end{pmatrix} \times \begin{pmatrix}-2\\2\\3\end{pmatrix} = \begin{pmatrix}5\\-10\\10\end{pmatrix}$

3 $\lambda^2 - \lambda - 12 = 0$, solutions $\lambda = 4$ and $\lambda = -3$.

For $\lambda = 4$, eigenvectors parallel to $\begin{pmatrix}1\\1\end{pmatrix}$.

For $\lambda = -3$, eigenvectors parallel to $\begin{pmatrix}-2\\5\end{pmatrix}$.

Exercise 12A

1 **a** The magnitude is $\sqrt{6^2 + 3^2} = 6.71$ N

The direction is $\tan^{-1}\frac{3}{6} = 26.6$

b The magnitude is $\sqrt{3^2 + 4^2} = 5$ N

The direction is $180 - \tan^{-1}\frac{-3}{4} = 126.9$

2 A vector of magnitude a and angle θ can be expressed as the column vector $\begin{pmatrix}a\cos\theta\\a\sin\theta\end{pmatrix}$.

a $\begin{pmatrix}7\cos 45\\7\sin 45\end{pmatrix}\text{ms}^{-1} = \begin{pmatrix}4.95\\4.95\end{pmatrix}\text{ms}^{-1}$

b Bearing of 330° from N is 120° anticlockwise from the x-axis:

$\begin{pmatrix}-12\cos 60\\12\sin 60\end{pmatrix}\text{ms}^{-1} = \begin{pmatrix}-6\\10.4\end{pmatrix}\text{ms}^{-1}$

3 The total force is $\begin{pmatrix}2\\4\end{pmatrix} + \begin{pmatrix}-1\\2\end{pmatrix} + \begin{pmatrix}-2\\0\end{pmatrix}\text{ms}^{-2} = \begin{pmatrix}-1\\6\end{pmatrix}\text{ms}^{-2}$

The magnitude is $\sqrt{(-1)^2 + 6^2}\text{ms}^{-2} = 6.08\text{ms}^{-2}$

The direction is $\tan^{-1}\frac{6}{-1} = 99.5°$ (quadrant II)

4 **a** The direction of the torque is perpendicular to the force F and displacement r.

b The torque τ is given by $\tau = \begin{pmatrix}3\\4\\0\end{pmatrix} \times \begin{pmatrix}2\\1\\0\end{pmatrix} = \begin{pmatrix}0\\0\\-5\end{pmatrix}$. The magnitude is 5 Nm.

5 **a** The resistance force is $\begin{pmatrix}-70\\0\end{pmatrix}$ N.

The force from the first dog is $\begin{pmatrix}150\cos 10\\150\sin 10\end{pmatrix} = \begin{pmatrix}147.72\\26.05\end{pmatrix}$N.

The force from the second dog is $\begin{pmatrix}D\cos(15)\\-D\sin(15)\end{pmatrix} = \begin{pmatrix}0.97D\\-0.26D\end{pmatrix}$N

b The resultant force is $\begin{pmatrix}-70\\0\end{pmatrix} + \begin{pmatrix}147.72\\26.05\end{pmatrix} + \begin{pmatrix}0.97D\\-0.26D\end{pmatrix} = \begin{pmatrix}77.72+0.97D\\26.05-0.26D\end{pmatrix}$N

c If the sled is travelling in the correct direction then the perpendicular component of the resultant force is 0:

$$\therefore 26.05 - 0.26D = 0 \Rightarrow D = \frac{26.05}{0.26} = 100.19\text{ N}$$

d If the sled travels in the correct direction then D = 100.19 and

$$F = \begin{pmatrix}77.72+0.97D\\0\end{pmatrix} = \begin{pmatrix}174.90\\0\end{pmatrix}\text{ N}$$

$F = ma$ where m = 60 kg, so $a = \frac{1}{60}\begin{pmatrix}174.90\\0\end{pmatrix} = \begin{pmatrix}2.92\\0\end{pmatrix}\text{ms}^{-2}$

The acceleration is 2.92 m s^{-2} directly forwards.

Exercise 12B

1 **a** Let $a = \begin{pmatrix}1\\4\end{pmatrix}$ and $b = \begin{pmatrix}-1\\2\end{pmatrix}$

Compute:

$$a \cdot b = (1 \times (-1)) + (4 \times 2) = 7$$

$$a \times b = \begin{pmatrix}0\\0\\6\end{pmatrix}$$

$$|b| = \sqrt{(-1)^2 + 2^2} = \sqrt{5}$$

The component of $\begin{pmatrix}1\\4\end{pmatrix}$ in the direction of $\begin{pmatrix}-1\\2\end{pmatrix}$ is $\frac{7}{\sqrt{5}} = 3.13$

The component of $\begin{pmatrix}1\\4\end{pmatrix}$ perpendicular to $\begin{pmatrix}-1\\2\end{pmatrix}$ within the plane defined by the two vectors is

$\frac{6}{\sqrt{5}} = 2.68$

b Let $a = \begin{pmatrix}2\\0\\5\end{pmatrix}$ and $b = \begin{pmatrix}2\\2\\-1\end{pmatrix}$

Compute:

$$a \cdot b = (2\times 2) + (0\times 2) + (5\times(-1)) = -1$$

$$a \times b = \begin{pmatrix}-10\\12\\4\end{pmatrix}$$

$$|b| = \sqrt{2^2 + 2^2 + (-1)^2} = 3$$

The component of $\begin{pmatrix}2\\0\\5\end{pmatrix}$ in the direction of $\begin{pmatrix}2\\2\\-1\end{pmatrix}$ is $\frac{-1}{3} = -0.33$

The component of $\begin{pmatrix}2\\0\\5\end{pmatrix}$ perpendicular to $\begin{pmatrix}2\\2\\-1\end{pmatrix}$ within the plane defined by the two vectors is

$$\frac{\sqrt{(-10)^2 + 12^2 + 4^2}}{3} = 5.38$$

2 a The resultant force F is $\begin{pmatrix}34\\18\\7\end{pmatrix}$ N.

If $b = \begin{pmatrix}12\\5\\0\end{pmatrix}$ is the direction of the rail then the component of the resultant force in the direction of the rail is $\frac{F\cdot b}{|b|} = \frac{(34\times 12)+(18\times 5)+(7\times 0)}{\sqrt{12^2+5^2+0^2}} = 38.31$ N

b If a is the acceleration of the truck of mass m = 150 kg then the acceleration along the rail is

$\frac{a\cdot b}{|b|} = \frac{1}{m}\cdot\frac{(ma)\cdot b}{|b|} = \frac{1}{150}\cdot\frac{F\cdot b}{|b|} = 0.255$ m s^{-2} by Newton's second law ($F = ma$).

3 a The velocity v of the current is $k\begin{pmatrix}3\\4\end{pmatrix}$ where

$\left|k\begin{pmatrix}3\\4\end{pmatrix}\right| = 10$, that is $k = 2$, so that $v = \begin{pmatrix}6\\8\end{pmatrix}$ km h^{-1}.

b If $b = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is the direction of the boat then the component of the current acting in the direction the boat is travelling is

$\frac{v \cdot b}{|b|} = \frac{(6 \times 1) + (8 \times 1)}{\sqrt{1^2 + 1^2}} = 9.90 \text{ km h}^{-1}$.

4 **a** Since a and n are perpendicular, $a \cdot n = 0$

As n is a unit vector, $n \cdot n = |n|^2 = 1$

We write $b_i = a + kn$ for some scalar k and try to find k

Now $b_i \cdot n = (a + kn) \cdot n = a \cdot n + kn \cdot n = k$

Thus, $b_i = a + (b_i \cdot n)n$

b Observe that $b_i + b_r = 2a \Rightarrow b_r = 2a - b_i$

Since $a = b_i - kn$, then $b_r = 2(b_i - kn) - b_i = b_i - 2kn = b_i - 2(b_i \cdot n)n$

c The normal vector $\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$ has magnitude $\sqrt{2^2 + 1^2 + (-2)^2} = 3$.

So the unit normal vector $n = \frac{1}{3}\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$

Also, $b_i = \begin{pmatrix} 0 \\ -1 \\ 4 \end{pmatrix}$

Now $k = b_i \cdot n = -3$ and so the direction b_r of the reflected ray is given by

$b_r = b_i - 2kn = \begin{pmatrix} 0 \\ -1 \\ 4 \end{pmatrix} + 2\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix}$

Exercise 12C

1 **a** When $t = 0$, $r = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$ m is the initial displacement.

b $v = \frac{dr}{dt} = \begin{pmatrix} 4 \\ 6t - 1 \end{pmatrix}$ m s^{-1}

The initial velocity is at $t = 0$, which gives $v = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$ m s^{-1} and the initial speed $|v| = 4.12$ ms^{-1}.

c $a = \frac{dv}{dt} = \begin{pmatrix} 0 \\ 6 \end{pmatrix}$ m s^{-2}, which has constant magnitude of 6 m s^{-2}.

2 **a** Integrating $r = (2t^2 - t + c_1)i + (5t - t^2 + c_2)j$ m.

The initial displacement at t = 0 is $2i - j$, giving $c_1 = 2, c_2 = -1$

Thus, the displacement at time t is $r = (2t^2 - t + 2)i + (5t - t^2 - 1)j$ m.

b When t = 2 s, $r = 8i + 5j$ m and so the distance of the particle P from O is $\sqrt{8^2 + 5^2} = 9.43$ m.

3 **a** Integrating, $v = \begin{pmatrix} 2t + c_1 \\ t + c_2 \end{pmatrix}$ m s^{-1}, where c_1, c_2 are constants.

The initial velocity at t = 0 is $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ m s^{-1} giving $c_1 = 2, c_2 = 3$.

Thus, the velocity of P at time t is $v = \begin{pmatrix} 2t + 2 \\ t + 3 \end{pmatrix}$ m s^{-1}.

b Integrating, $r = \begin{pmatrix} t^2 + 2t + d_1 \\ \frac{1}{2}t^2 + 3t + d_2 \end{pmatrix}$ m, where d_1, d_2 are constants.

The initial displacement of P is $\begin{pmatrix} 4 \\ -1 \end{pmatrix}$ m giving $d_1 = 4, d_2 = -1$.

Thus, the displacement of P at time t is $r = \begin{pmatrix} t^2 + 2t + 4 \\ \frac{1}{2}t^2 + 3t - 1 \end{pmatrix}$ m.

4 **a** $v = (2t - 1)i + 3t^2 j$

The particle is moving parallel to j when $2t - 1 = 0$, i.e. when $t = \frac{1}{2}$.

b When $t = 2$, $v = 3i + 12j$. The angle ϑ that the velocity vector makes with the vector i satisfies $\tan\theta = \frac{12}{3} = 4 \Rightarrow \theta = 76.0°$.

c Integrating, $r = (t^2 - t + c_1)i + (t^3 + c_2)j$, where c_1, c_2 are constants.

The initial displacement $2i - 3j$ giving $c_1 = 2, c_2 = -3$.

Thus, the displacement of the particle at time t is $r = (t^2 - t + 2)i + (t^3 - 3)j$.

d Solving $(t^2 - t + 2)^2 + (t^3 - 3)^2 - 256 = 0$ on the GDC gives t = 2.61.

5 **a** By integrating, $v = \begin{pmatrix} 3t^2 + c_1 \\ 2t + c_2 \end{pmatrix}$ m s^{-1} where c_1, c_2 are constants.

When $t = 1, v = \begin{pmatrix} 7 \\ 14 \end{pmatrix}$ m s^{-1} and so

$3 + c_1 = 7 \Rightarrow c_1 = 4$
$2 + c_2 = 14 \Rightarrow c_2 = 12$

Thus, $v = \begin{pmatrix} 3t^2 + 4 \\ 2t + 12 \end{pmatrix}$ m s^{-1}.

b The direction of motion of the aircraft is at 45° to the horizontal when $3t^2 + 4 = 2t + 12$ or $(3t + 4)(t - 2) = 0$.

Thus, $t = 2$ s or $t = -\frac{4}{3}$ s.

The value $t = -\frac{4}{3}$ s is impossible since $t \geq 0$. Thus, the direction of motion of the aircraft is at 45° to the horizontal when $t = 2$ s.

6 **a** Integrating, $r_P = \begin{pmatrix} 2t^2 - 2.5t + c_1 \\ t^3 + c_2 \end{pmatrix}$ m where c_1 and c_2 are constants.

When $t = 0$ s, $r_p = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$ m $\Rightarrow r_p = \begin{pmatrix} 2t^2 - 2.5t + 2 \\ t^3 \end{pmatrix}$ m.

b By integrating, $v_Q = \begin{pmatrix} c_3 \\ 4t + c_4 \end{pmatrix}$ m s^{-1}, where c_3 and c_4 are constants.

The vector $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ has a magnitude of 5, so the initial velocity of Q is $\begin{pmatrix} 6 \\ 8 \end{pmatrix}$ m s^{-1}, so $v_Q = \begin{pmatrix} 6 \\ 4t + 8 \end{pmatrix}$ m s^{-1}.

c Integrating, $r_Q = \begin{pmatrix} 6t + c_5 \\ 2t^2 + 8t + c_6 \end{pmatrix}$ m, where c_5 and c_6 are constants.

When $t = 0$ s, $r_Q = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ m $\Rightarrow r_Q = \begin{pmatrix} 6t \\ 2t^2 + 8t \end{pmatrix}$ m.

The two particles collide at time t if and only if $r_P = r_Q$ at time t.

From the j components, $t^3 = 2t^2 + 8t$, so either $t = 0$ or $t^2 = 2t + 8$.

$t = 0$ s is not a solution to $r_P = r_Q$ since the initial displacements are not the same.

Thus, $t^2 = 2t + 8 \Rightarrow t^2 - 2t - 8 = 0 \Rightarrow (t - 4)(t + 2) = 0 \Rightarrow t = 4$ or $t = -2$.

Time cannot be negative, and so the particles collide at $t = 4$ s.

To confirm, for the i components at $t = 4$ s, $2t^2 - 2.5t + 2 = 24$ and $6t = 24$, so the i components also coincide at $t = 4$ s.

7 Differentiating r gives $v = 4i + (2t - 3)j$ km h^{-1}.

The speed is $\sqrt{4^2 + (2t - 3)^2}$ km h^{-1}, which is a minimum when $4^2 + (2t - 3)^2$ is.

This clearly happens when $t = 1.5$ hr and for this t, the speed is 4 km h^{-1}.

Exercise 12D

1 **a** $a = \begin{pmatrix} 0 \\ -g \end{pmatrix} = \begin{pmatrix} 0 \\ -9.81 \end{pmatrix}$ m s^{-2}.

b **i** $v = \begin{pmatrix} c_1 \\ c_2 - gt \end{pmatrix}$ m s^{-1}.

The initial velocity is $\begin{pmatrix} 5 \\ 2 \end{pmatrix}$ m s^{-1} and so $v = \begin{pmatrix} 5 \\ 2 - gt \end{pmatrix} = \begin{pmatrix} 5 \\ 2 - 9.81t \end{pmatrix}$ m s^{-1}.

ii $r = \begin{pmatrix} 5t + c_3 \\ 2t - \frac{g}{2}t^2 + c_4 \end{pmatrix}$ m.

The initial displacement is $\begin{pmatrix} 0 \\ 1.5 \end{pmatrix}$ m and so $r = \begin{pmatrix} 5t \\ 2t - \frac{g}{2}t^2 + 1.5 \end{pmatrix} = \begin{pmatrix} 5t \\ 2t - 4.905t^2 + 1.5 \end{pmatrix}$ m.

c The particle hits the ground when $2t - \frac{g}{2}t^2 + 1.5 = 0$, i.e. when $\frac{g}{2}t^2 - 2t - 1.5 = 0$.

Since $t > 0$, we have $t = 0.793$ s

The horizontal distance travelled is then $= 3.97$ m.

2 **a** $v = 4i + (2 - gt)j$ m s^{-1}

b Integrating, $r = (4t + c_1)i + \left(2t - \frac{g}{2}t^2 + c_2\right)j$ m

Since $r = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ when $t = 0$, $c_1 = c_2 = 0$.

Thus, $r = (4t)i + \left(2t - \frac{g}{2}t^2\right)j$ m.

c The stone hits the ground when $2t - \frac{g}{2}t^2 = -50$ m, or $\frac{g}{2}t^2 - 2t - 50 = 0$.

As $t > 0$, $t = 3.40$ s

d The horizontal distance travelled by time t is $4t$ m.

Thus, the distance from the base of the cliff after hitting the ground is $4 \times 3.40 = 13.6$ m.

3 **a** The initial velocity is $\begin{pmatrix} 14.7\cos(30) \\ 14.7\sin(30) \end{pmatrix} = \begin{pmatrix} 12.7 \\ 7.35 \end{pmatrix}$ m s^{-1}.

b $a = \begin{pmatrix} 0 \\ -g \end{pmatrix}$ m s^{-2} giving $v = \begin{pmatrix} c_1 \\ -gt + c_2 \end{pmatrix}$ m s^{-1}.

Using the initial velocity in part **a**, $v = \begin{pmatrix} 12.7 \\ 7.35 - gt \end{pmatrix} = \begin{pmatrix} 12.7 \\ 7.35 - 9.81t \end{pmatrix}$ m s^{-1}.

c The maximum height is reached when $7.35 - gt = 0$, i.e. $t = 0.749$ s.

d Integrating v, $r = \begin{pmatrix} 12.7t \\ 7.35t - \frac{g}{2}t^2 \end{pmatrix} = \begin{pmatrix} 12.7t \\ 7.35t - 4.905t^2 \end{pmatrix}$ m.

e Using the value of the time t at which the maximum height is reached from part **c**, the maximum height is $7.35t - \frac{g}{2}t^2 = 2.75$ m.

f The particle hits the ground at time $t > 0$ when $7.35t - \frac{g}{2}t^2 = 0 \Rightarrow t = 1.50$ s

The horizontal distance travelled is $12.7t$ m, which is 19.1 m.

g The speed of the particle at time t is $\sqrt{12.7^2 + (7.35 - 9.81t)^2}$ m s^{-1}.

This is a minimum when $7.35 - 9.81t = 0$, and for this value of t, the minimum speed is 12.7 m s^{-1}.

4 Suppose that we project at an initial speed of u m s^{-1}, so that the initial velocity is $\begin{pmatrix} u\cos(30) \\ u\sin(30) \end{pmatrix}$ m s^{-1} and the velocity at time is $v = \begin{pmatrix} u\cos(30) \\ u\sin(30) - gt \end{pmatrix}$ m s^{-1}.

By integrating, the displacement is $r = \begin{pmatrix} ut\cos(30) \\ ut\sin(30) - \frac{g}{2}t^2 \end{pmatrix}$ m.

The object hits the ground when $ut\sin(30) - \frac{g}{2}t^2 = 0$ and we are told this happens when $t = 6$ s.

Thus, $3u - 18g = 0 \Rightarrow u = 6g = 58.9$ m s^{-1}.

5 **a** The initial velocity is $\begin{pmatrix} 10\cos(45) \\ 10\sin(45) \end{pmatrix} = \begin{pmatrix} 5\sqrt{2} \\ 5\sqrt{2} \end{pmatrix}$ or $\begin{pmatrix} 7.07\ldots \\ 7.07\ldots \end{pmatrix}$ m s^{-1}.

The velocity at time is $v = \begin{pmatrix} 7.07 \\ 7.07 - 9.81t \end{pmatrix}$ m s^{-1}.

By integrating, the displacement is $r = \begin{pmatrix} 7.07t + c_1 \\ 7.07t - 4.905t^2 + c_2 \end{pmatrix}$ m.

When $t = 0$, $r = \begin{pmatrix} 0 \\ 1.5 \end{pmatrix}$ m. Thus, $r = \begin{pmatrix} 7.07t \\ 7.07t - 9.81t^2 + 1.5 \end{pmatrix}$.

b The particle has travelled a horizontal distance of 10 m at time t for which $7.07t = 10$, i.e. $t = 1.41\ldots$s.

Let h be the height of the wall. The wall is exactly 10 m away from the point of projection and so $h = 7.07t - 4.095t^2 + 1.5$ where $t = 1.41\ldots$, giving $h = 11.5 - 9.1 = 1.69$ m.

c The ball hits the ground when $7.07t - 4.905t^2 + 1.5 = 0$

Solving this with $t > 0$ gives $t = 1.63$s.

The horizontal distance travelled is $7.07t = 11.5$ m.

6 **a** The initial velocity vector is $\begin{pmatrix} 50\cos(30) \\ 50\sin(30) \end{pmatrix} = \begin{pmatrix} 43.3 \\ 25 \end{pmatrix}$ m s^{-1}.

b The velocity vector is $\begin{pmatrix} 43.3 \\ 25 - 9.81t \end{pmatrix}$ m s^{-1}, so $r = \begin{pmatrix} 43.3t + c_1 \\ 25t - 4.905t^2 + c_2 \end{pmatrix}$ m.

The displacement is measured from the point of projection, so $c_1 = c_2 = 0$ and

$r = \begin{pmatrix} 43.3t \\ 25t - 4.905t^2 \end{pmatrix}$ m.

c Measuring from the initial point of displacement, we want to work out how long the particle is a height of at least 22 m above the point of displacement.

This occurs if and only if $25t - 4.905t^2 > 22$,

Thus, the particle is at least 22 m above the ground for a time $3.965... - 1.1309... \approx 2.83$ s.

d The particle hits the ground when $25t - 4.905t^2 = -2 \Rightarrow 4.905t^2 - 25t - 2 = 0$, so $t = 5.18$ s as t > 0.

The horizontal distance is 224 m.

Exercise 12E

1 **a** $a = \frac{dv}{dt} = \begin{pmatrix} 4e^{2t} \\ 2e^{2t} \end{pmatrix}$.

b Integrating, $r = \begin{pmatrix} e^{2t} + c_1 \\ \frac{1}{2}e^{2t} + 2t + c_2 \end{pmatrix}$.

When t = 0, $r = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow c_1 = -1$ and $c_2 = -\frac{1}{2}$.

Thus, $r = \begin{pmatrix} e^{2t} - 1 \\ \frac{1}{2}e^{2t} + 2t - \frac{1}{2} \end{pmatrix}$.

2 $v = \begin{pmatrix} 5e^{2t} - t + c_1 \\ 2e^{2t} + 2t + c_2 \end{pmatrix}$ by integration.

When t = 0, the particle is at rest and so $v = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

This gives $c_1 = -5$ and $c_2 = -2$.

$v = \begin{pmatrix} 5e^{2t} - t - 5 \\ 2e^{2t} + 2t - 2 \end{pmatrix}$, and by integrating again $r = \begin{pmatrix} \frac{5}{2}e^{2t} - \frac{1}{2}t^2 - 5t + c_3 \\ e^{2t} + t^2 - 2t + c_4 \end{pmatrix}$.

When t = 0, $r = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and so $c_3 = -\frac{5}{2}$ and $c_4 = -1$.

Thus, $r = \begin{pmatrix} \frac{5}{2}e^{2t} - \frac{1}{2}t^2 - 5t - \frac{5}{2} \\ e^{2t} + t^2 - 2t - 1 \end{pmatrix}$.

3 **a** $v = \begin{pmatrix} 4\sin(2t) + c_1 \\ -4\cos(2t) + c_2 \end{pmatrix}$ by integrating.

When $t = 0$, $v = \begin{pmatrix} 0 \\ -4 \end{pmatrix}$ and so $c_1 = c_2 = 0$.

By integrating again, $r = \begin{pmatrix} -2\cos(2t) + c_3 \\ -2\sin(2t) + c_4 \end{pmatrix}$.

When $t = 0$, $r = \begin{pmatrix} -2 \\ 0 \end{pmatrix}$ and so $c_3 = c_4 = 0$.

Thus, $r = \begin{pmatrix} -2\cos(2t) \\ -2\sin(2t) \end{pmatrix}$.

b The distance from the origin is

$|r| = \sqrt{(-2\cos(2t))^2 + (-2\sin(2t))^2} = \sqrt{4(\sin^2(2t) + \cos^2(2t))} = \sqrt{4} = 2$.

c The particle moves anticlockwise in a circle of radius 2 and centre $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

d $k = -4$.

4 **a** $v = \frac{dr}{dt} = \begin{pmatrix} 2e^{2t} \\ 6e^{2t} \end{pmatrix}$

b $a = \frac{dv}{dt} = \begin{pmatrix} 4e^{2t} \\ 12e^{2t} \end{pmatrix}$

c $k = 4$

5 **a** The distance from $\begin{pmatrix} 2 \\ 5 \end{pmatrix}$ is

$\left| r - \begin{pmatrix} 2 \\ 5 \end{pmatrix} \right| = \left| \begin{pmatrix} e^{-t}\cos 4t \\ e^{-t}\sin 4t \end{pmatrix} \right| = \sqrt{(e^{-t}\cos 4t)^2 + (e^{-t}\sin 4t)^2} = \sqrt{e^{-2t}(\cos^2 4t + \sin^2 4t)} = \sqrt{e^{-2t}} = e^{-t}$.

b The particle spirals in towards the point $\begin{pmatrix} 2 \\ 5 \end{pmatrix}$.

c $v = \frac{dr}{dt} = \begin{pmatrix} -4e^{-t}\sin 4t - e^{-t}\cos 4t \\ 4e^{-t}\cos 4t - e^{-t}\sin 4t \end{pmatrix}$ via the chain and product rules.

Write $s = \sin 4t$ and $c = \cos 4t$ to simplify the notation.

The speed is

$$|v| = e^{-t}\left|\begin{pmatrix}-4s-c\\4c-s\end{pmatrix}\right|$$
$$= e^{-t}\sqrt{(-4s-c)^2+(4c-s)^2}$$
$$= e^{-t}\sqrt{16s^2+8sc+c^2+16c^2-8sc+s^2}\text{ s}$$
$$= e^{-t}\sqrt{17s^2+17c^2}$$
$$= e^{-t}\sqrt{17}$$

d The initial speed is $e^{-0}\sqrt{17}=\sqrt{17}$.

This is reduced by half when $e^{-t}\sqrt{17}=\frac{1}{2}\sqrt{17} \Rightarrow e^{-t}=\frac{1}{2}$, so $t=\ln(2)=0.693$ s.

6 a Every 6 seconds the particle travels 2π radians.

Thus, in 1 second the particle travels $\frac{\pi}{3}$ radians.

Using trigonometry with angle $\frac{\pi}{3}$ radians and radius 4:

$$r = \begin{pmatrix}4\sin\left(\frac{\pi}{3}t\right)\\-4\cos\left(\frac{\pi}{3}t\right)\end{pmatrix}$$

b The particle moves in the same motion but with centre (5, 4), so add 5 to x and 4 to y:

$$r = \begin{pmatrix}4\sin\left(\frac{\pi}{3}t\right)+5\\-4\cos\left(\frac{\pi}{3}t\right)+4\end{pmatrix}$$

c $v = \begin{pmatrix}\frac{4\pi}{3}\cos\left(\frac{\pi}{3}t\right)\\\frac{4\pi}{3}\sin\left(\frac{\pi}{3}t\right)\end{pmatrix}$

Exercise 12F

1 a i $M=\begin{pmatrix}2&2\\5&-1\end{pmatrix}$

$|M-\lambda I| = \begin{vmatrix}2-\lambda&2\\5&-1-\lambda\end{vmatrix} = (\lambda-4)(\lambda+3)$ so $\lambda_1=4, \lambda_2=-3$.

Two corresponding eigenvectors are $\begin{pmatrix}1\\1\end{pmatrix}$ and $\begin{pmatrix}-2\\5\end{pmatrix}$.

Thus, $\boldsymbol{x} = Ae^{-3t}\begin{pmatrix}-2\\5\end{pmatrix}+Be^{4t}\begin{pmatrix}1\\1\end{pmatrix}$.

By the initial conditions, $\begin{pmatrix} -6 \\ 22 \end{pmatrix} = \begin{pmatrix} -2A+B \\ 5A+B \end{pmatrix} \Rightarrow A = 4, B = 2$, so $\boldsymbol{x} = 4e^{-3t}\begin{pmatrix} -2 \\ 5 \end{pmatrix} + 2e^{4t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

ii $M = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$

$|M - \lambda I| = \begin{vmatrix} 1-\lambda & 2 \\ 0 & 3-\lambda \end{vmatrix} = (\lambda - 3)(\lambda - 1)$, so $\lambda_1 = 1, \lambda_2 = 3$.

Two corresponding eigenvectors are $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Thus $\boldsymbol{x} = Ae^{t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + Be^{3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

By the initial conditions, $\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} A+B \\ B \end{pmatrix} \Rightarrow A = 2, B = 1$, so $\boldsymbol{x} = 2e^{t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + e^{3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

iii $M = \begin{pmatrix} -2 & 2 \\ -1 & -5 \end{pmatrix}$

$|M - \lambda I| = \begin{vmatrix} -2-\lambda & 2 \\ -1 & -5-\lambda \end{vmatrix} = (\lambda + 3)(\lambda + 4)$, so $\lambda_1 = -3, \lambda_2 = -4$.

Two corresponding eigenvectors are $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

Thus $\boldsymbol{x} = Ae^{-3t}\begin{pmatrix} 2 \\ -1 \end{pmatrix} + Be^{-4t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

By the initial conditions, $\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2A+B \\ -A-B \end{pmatrix} \Rightarrow A = 3, B = -4$, so $\boldsymbol{x} = 3e^{-3t}\begin{pmatrix} 2 \\ -1 \end{pmatrix} - 4e^{-4t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

iv $M = \begin{pmatrix} 1 & 2 \\ 3 & -4 \end{pmatrix}$

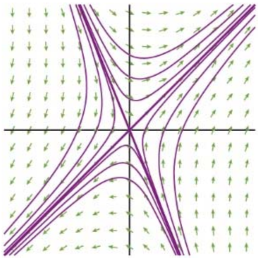
$|M - \lambda I| = \begin{vmatrix} 1-\lambda & 2 \\ 3 & -4-\lambda \end{vmatrix} = (\lambda - 2)(\lambda + 5)$, so $\lambda_1 = -5, \lambda_2 = 2$.

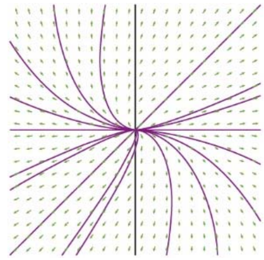
Two corresponding eigenvectors are $\begin{pmatrix} -1 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

Thus $\boldsymbol{x} = Ae^{-5t}\begin{pmatrix} -1 \\ 3 \end{pmatrix} + Be^{2t}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

By the initial conditions, $\begin{pmatrix} 1 \\ 11 \end{pmatrix} = \begin{pmatrix} -A+2B \\ 3A+B \end{pmatrix} \Rightarrow A = 3, B = 2$, so $\boldsymbol{x} = 3e^{-5t}\begin{pmatrix} -1 \\ 3 \end{pmatrix} + 2e^{2t}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

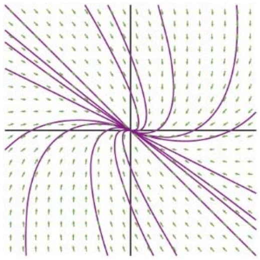
b **i** $y = x, y = -\frac{5}{2}x$ **ii** $y = x, y = 0$

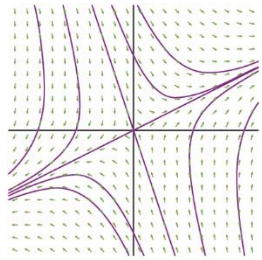




iii $y = -x, y = -\frac{1}{2}x$

iv $y = -3x, y = \frac{1}{2}x$





2 **a** D **b** A **c** C **d** B

3 **a** $M = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$

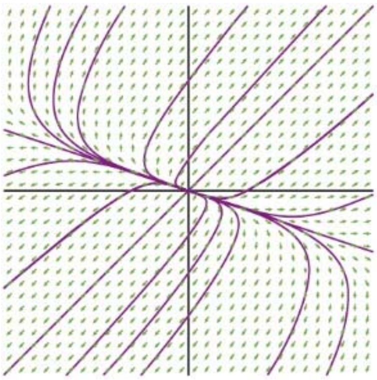
$|M - \lambda I| = \begin{vmatrix} 2-\lambda & 3 \\ 1 & 4-\lambda \end{vmatrix} = (\lambda - 1)(\lambda - 5)$, so $\lambda_1 = 1, \lambda_2 = 5$.

Two corresponding eigenvectors are $\begin{pmatrix} -3 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Thus, $\boldsymbol{x} = Ae^{t}\begin{pmatrix} -3 \\ 1 \end{pmatrix} + Be^{5t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

b The asymptotes are $y = x$ and $y = -\frac{1}{3}x$.

The asymptotic behaviour is that $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is an equilibrium point. Trajectories on the line $y = -\frac{1}{3}x$ stay on that line and move away from $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and all other trajectories move away from $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and towards the line $y = x$.



c By the initial conditions, $\begin{pmatrix}-9\\-1\end{pmatrix} = A\begin{pmatrix}-3\\1\end{pmatrix} + B\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}-3A+B\\A+B\end{pmatrix} \Rightarrow A = 2, B = -3$, so

$\boldsymbol{x} = 2e^{t}\begin{pmatrix}-3\\1\end{pmatrix} - 3e^{5t}\begin{pmatrix}1\\1\end{pmatrix}$.

4 a $M = \begin{pmatrix}3 & 1\\2 & -1\end{pmatrix}$

$|M - \lambda I| = \begin{vmatrix}3-\lambda & 1\\2 & -1-\lambda\end{vmatrix} = \lambda^2 - 2\lambda - 5$, so $\lambda = \dfrac{2 \pm \sqrt{4+20}}{2} = 1 \pm \sqrt{6}$.

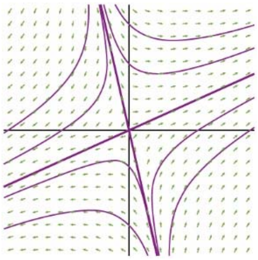
Two corresponding eigenvectors are $\begin{pmatrix}1\\-2\pm\sqrt{6}\end{pmatrix}$.

Thus, $\boldsymbol{x} = Ae^{(1-\sqrt{6})t}\begin{pmatrix}1\\-2-\sqrt{6}\end{pmatrix} + Be^{(1+\sqrt{6})t}\begin{pmatrix}1\\-2+\sqrt{6}\end{pmatrix}$, or to 3 s.f.,

$\boldsymbol{x} = Ae^{-1.45t}\begin{pmatrix}1\\-4.45\end{pmatrix} + Be^{3.45t}\begin{pmatrix}1\\0.449\end{pmatrix}$

b The asymptotes are $y = -4.45x$ and $y = 0.449x$.

The asymptotic behaviour is that $\begin{pmatrix}0\\0\end{pmatrix}$ is an equilibrium point. Trajectories on the line $y = -4.45x$ stay on that line and move towards $\begin{pmatrix}0\\0\end{pmatrix}$ and all other trajectories move away from $\begin{pmatrix}0\\0\end{pmatrix}$ and towards the line $y = 0.449x$.



c By the initial conditions, $\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}A+B\\(-2-\sqrt{6})A + (\sqrt{6}-2)B\end{pmatrix}$.

These equations have exact solution $A = \dfrac{2-\sqrt{6}}{4}, B = \dfrac{2+\sqrt{6}}{4}$, or to 3 s.f., $A = -0.112, B = 1.11$.

This gives $\boldsymbol{x} = -0.112e^{-1.45t}\begin{pmatrix}1\\-4.45\end{pmatrix} + 1.11e^{3.45t}\begin{pmatrix}1\\0.449\end{pmatrix}$.

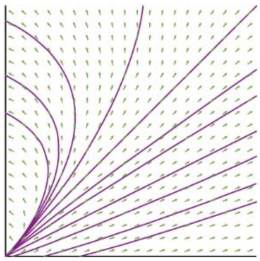
5 a When the initial conditions are $x = 0, y = 0$, then $\dfrac{dx}{dt} = \dfrac{dy}{dt} = 0$

So the trajectory remains at $\begin{pmatrix}0\\0\end{pmatrix}$.

b (0,0) is a stable equilibrium when both eigenvalues are negative.

c Fred's statement is false, and is a sufficient condition for a saddle point instead. We need every point near the equilibrium to tend towards it and not just some of them. What Fred describes occurs if there is one positive and one negative eigenvalue.

6 a



b i If $x = 0$, then replace the system by $\frac{dx}{dt} = 0, \frac{dy}{dt} = 0.1y$ so that $y = Be^{0.1t}$ and $x = 0$.

ii If $y = 0$, then replace the system by $\frac{dx}{dt} = 0.4x, \frac{dy}{dt} = 0$ so that $x = Ae^{0.4t}$ and $y = 0$.

c From the phase portrait, we end up with the following ratios of X to Y in each case.

i 2:1 **ii** 2:1 **iii** 0:1 **iv** 1:1

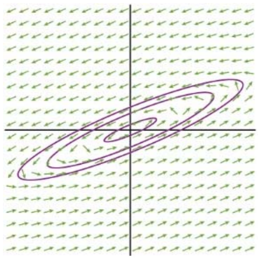
Exercise 12G

1 a i $M = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix}$

$|M - \lambda I| = \begin{vmatrix} 2-\lambda & -5 \\ 1 & -2-\lambda \end{vmatrix} = \lambda^2 + 1$, so $\lambda = \pm i$.

ii $\frac{dy}{dt} = 1 - (2 \times 0) = 1$ and $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{1}{2}$

iii

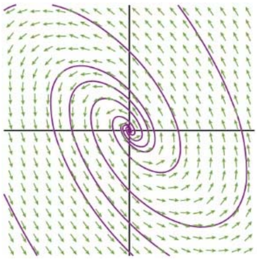


b i $M = \begin{pmatrix} -1 & -3 \\ 4 & 3 \end{pmatrix}$

$|M - \lambda I| = \begin{vmatrix} -1-\lambda & -3 \\ 4 & 3-\lambda \end{vmatrix} = \lambda^2 - 2\lambda + 9$, so $\lambda = \frac{2 \pm \sqrt{4 - 4 \times 9}}{2} = 1 \pm 2\sqrt{2}i$

ii $\frac{dy}{dt} = 4$ and $\frac{dy}{dx} = \frac{4}{-1} = -4$.

iii

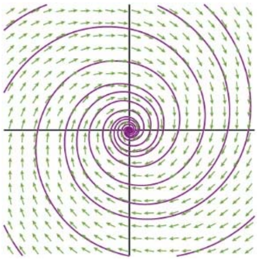


c **i** $M = \begin{pmatrix} -0.2 & 1 \\ -1 & -0.2 \end{pmatrix}$

$|M - \lambda I| = \begin{vmatrix} -0.2-\lambda & 1 \\ -1 & -0.2-\lambda \end{vmatrix} = (\lambda + 0.2)^2 + 1$, so $\lambda = -0.2 \pm i$.

ii $\frac{dy}{dt} = -1$ and $\frac{dy}{dx} = \frac{-1}{-0.2} = 5$.

iii

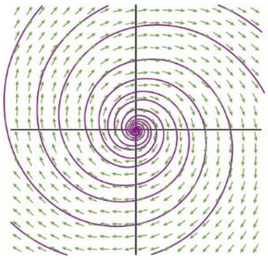


d **i** $M = \begin{pmatrix} 1 & 5 \\ -5 & 1 \end{pmatrix}$

$|M - \lambda I| = \begin{vmatrix} 1-\lambda & 5 \\ -5 & 1-\lambda \end{vmatrix} = (\lambda - 1)^2 + 25$, so $\lambda = 1 \pm 5i$.

ii $\frac{dy}{dt} = -5$ and $\frac{dy}{dx} = \frac{-5}{1} = -5$.

iii

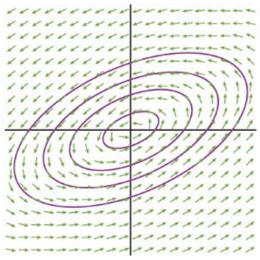


e **i** $M = \begin{pmatrix} 3 & -9 \\ 4 & -3 \end{pmatrix}$

$|M - \lambda I| = \begin{vmatrix} 3-\lambda & -9 \\ 4 & -3-\lambda \end{vmatrix} = \lambda^2 + 27$, so $\lambda = \pm 3\sqrt{3}i$.

ii $\frac{dy}{dt} = 4$ and $\frac{dy}{dx} = \frac{4}{3}$.

iii

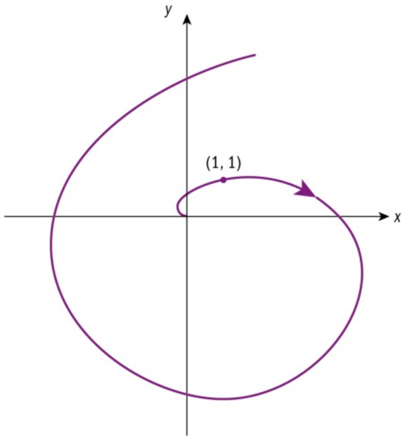


2 **a** We have $M = \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}$.

$|M - \lambda I| = \begin{vmatrix} 2-\lambda & 1 \\ -1 & 2-\lambda \end{vmatrix} = (\lambda - 2)^2 + 1$, so $\lambda = 2 \pm i$.

The real part is positive, so the satellite spirals away from the earth at $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ as claimed.

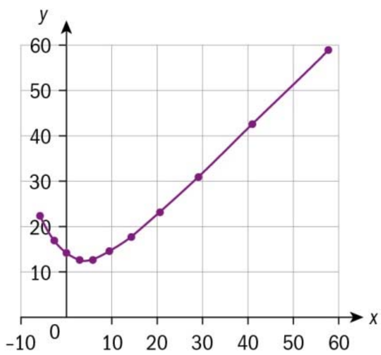
b



Exercise 12H

1 **a** **i** The formulae from the Euler method are $x_{n+1} = 1.2x_n + 0.2y_n, y_{n+1} = 0.5x_n + 0.9y_n$.

t	x_n	y_n
0	-6	22
0.1	-2.800	16.800
0.2	0	13.720
0.3	2.744	12.348
0.4	5.762	12.485
0.5	9.412	14.118
0.6	14.118	17.412
0.7	20.424	22.730
0.8	29.055	30.669
0.9	40.999	42.129
1.0	57.625	58.416

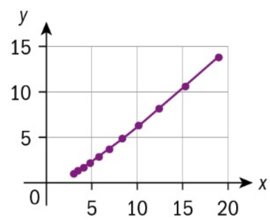


We can observe that $\frac{y_{10}}{x_{10}} = 1.01$ and when t = 1, $x = 108.8, y = 110.2$.

The Euler method severely underestimates the values of x and y.

ii The formulae from the Euler method are $x_{n+1} = 1.1x_n + 0.2y_n, y_{n+1} = 1.3y_n$.

t	x_n	y_n
0	3	1
0.1	3.500	1.300
0.2	4.110	1.690
0.3	4.859	2.197
0.4	5.784	2.856
0.5	6.934	3.713
0.6	8.370	4.827
0.7	10.172	6.275
0.8	12.444	8.157
0.9	15.320	10.604
1.0	18.973	13.786



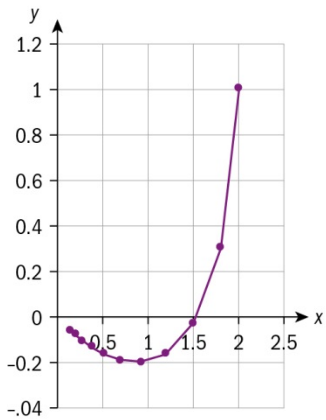
We can observe that $\frac{y_{10}}{x_{10}} = 0.73$ and when t = 1, $x = 25.52, y = 20.09$.

The Euler method underestimates the values of x and y and also has a slow rate of convergence of $\frac{y_n}{x_n}$ to 1.

iii The formulae from the Euler method are $x_{n+1} = 0.8x_n + 0.2y_n, y_{n+1} = -0.1x_n + 0.5y_n$.

t	x_n	y_n
0	2	1
0.1	1.800	0.300
0.2	1.500	-0.030
0.3	1.194	-0.165
0.4	0.922	-0.202
0.5	0.697	-0.193
0.6	0.519	-0.166

0.7	0.382	-0.135
0.8	0.279	-0.106
0.9	0.202	-0.081
1.0	0.145	-0.061

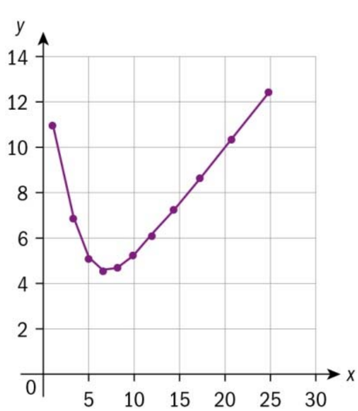


We can observe that $\frac{y_{10}}{x_{10}} = -0.42$ and when $t = 1, x = 0.23, y = -0.08$.

The Euler is misestimating the values of x and y, although the ratio of x:y is closer to the limiting value of $-\frac{1}{2}$ than for the exact values.

iv The formulae from the Euler method are $x_{n+1} = 1.1x_n + 0.2y_n, y_{n+1} = 0.3x_n + 0.6y_n$.

t	x_n	y_n
0	1	11
0.1	3.300	6.900
0.2	5.010	5.130
0.3	6.537	4.581
0.4	8.107	4.710
0.5	9.860	5.258
0.6	11.897	6.113
0.7	14.309	7.237
0.8	17.188	8.635
0.9	20.633	10.337
1.0	24.764	12.392



We can observe that $\frac{y_{10}}{x_{10}} = 0.50$ and when $t = 1$, $x = 29.54, y = 14.84$.

The Euler method has some imprecision with the exact values of x, y, although the ratio converges to the limit of $\frac{1}{2}$ much, much faster than for the exact data.

2 **a** The formulae from the Euler method are $x_{n+1} = x_n(0.9 + 0.2y_n), y_{n+1} = y_n(1.2 - 0.1x_n)$.

t	x_n	y_n
0	1	1
0.1	1.100	1.100
0.2	1.232	1.199
0.3	1.404	1.291
0.4	1.626	1.368
0.5	1.909	1.419
0.6	2.260	1.432
0.7	2.681	1.395
0.8	3.161	1.300
0.9	3.666	1.149
1.0	4.142	0.958

b The formulae from the Euler method are

$x_{n+1} = x_n(1 + 0.2x_n - 0.1y_n), y_{n+1} = y_n(1 + 0.2x_n - 0.1y_n)$.

t	x_n	y_n
0	1	1
0.1	1.100	1.100
0.2	1.221	1.221
0.3	1.370	1.370
0.4	1.558	1.558
0.5	1.800	1.800
0.6	2.125	2.125
0.7	2.576	2.576
0.8	3.240	3.240
0.9	4.289	4.289
1.0	6.129	6.129

3 **a** The formulae from the Euler method are

$$t_{n+1} = t_n + 0.1$$
$$x_{n+1} = x_n + 0.1(-2t_nx_n + 3y_n^2)$$
$$y_{n+1} = y_n + 0.1(-3x_n^2 + 3t_ny_n)$$

n	t_n	x_n	y_n
0	0	-1	2
1	0.1	0.200	1.700
2	0.2	1.063	1.739
3	0.3	1.928	1.504
4	0.4	2.491	0.525

5	0.5	2.374	-1.274

b The formulae from the Euler method are

$$t_{n+1} = t_n + 0.1$$
$$x_{n+1} = x_n + 0.1\left(2t_n x_n + y_n\right)$$
$$y_{n+1} = y_n + 0.1\left(-3x_n^2 + t_n y_n\right)$$

n	t_n	x_n	y_n
0	0	0	1
1	0.1	0.100	1.000
2	0.2	0.202	1.007
3	0.3	0.311	1.015
4	0.4	0.431	1.016
5	0.5	0.567	1.001

Exercise 12I

1 **a** If $\frac{dP}{dt} = 0$ then either $P = 0$ or $Q = 2$

If $P = 0$ then $Q = 0$ from $\frac{dQ}{dt} = 0$

If $Q = 2$ then $\frac{dP}{dt} = 0 \Rightarrow P = 1$

The equilibrium points are $(P, Q) = (0, 0)$ and $(P, Q) = (1, 2)$.

b If $P = 0$ then $\frac{dP}{dt} = 0$ so $P = 0$ forever and $\frac{dQ}{dt} = Q \Rightarrow Q = Ae^t$, which is exponential growth.

c If $Q = 0$ then $\frac{dQ}{dt} = 0$ so $Q = 0$ forever and $\frac{dP}{dt} = -2P \Rightarrow P = Be^{-2t}$, which is exponential decay.

d The formulae from the Euler method are

$$P_{n+1} = 0.1P_n\left(8 + Q_n\right)$$
$$Q_{n+1} = 0.1Q_n(11 - P_n)$$
and
$$P_0 = 2, Q_0 = 2$$

t	P_n	Q_n
0	2	2
0.1	2.000	1.800
0.2	1.960	1.620
0.3	1.886	1.464
0.4	1.785	1.335
0.5	1.666	1.230
0.6	1.538	1.148
0.7	1.407	1.086
0.8	1.278	1.042

0.9	1.156	1.013
1.0	1.042	0.998

Hence, there are approximately 1042 prey and 998 predators.

2 a $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} = 0 \Rightarrow x - 2xy = 0$ and $1.8xy - 1.5y = 0$, respectively.

If $x = 0$ then $y = 0$ and vice-versa.

When $x \neq 0, 1 - 2y = 0 \Rightarrow y = 0.5$ and $y(1.8x - 1.5) = 0 \Rightarrow x = \frac{1.5}{1.8} = \frac{5}{6}$.

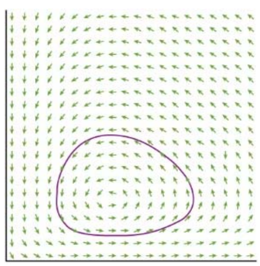
The non-zero equilibrium is 833 rabbits and 50 foxes.

b When $x = 0.75$ and $y = 1$, $\frac{dx}{dt} = -0.75$ and $\frac{dy}{dt} = -0.15$. Both populations are initially decreasing.

c i When $\frac{dx}{dt} = 0, x(1 - 2y) = 0$ so as $x \neq 0, y = \frac{1}{2}$, or 50 foxes.

ii When $\frac{dy}{dt} = 0, y(1.8x - 1.5) = 0$ so as $y \neq 0, x = \frac{5}{6}$, or 833 rabbits.

d



It can be observed from the (unlabelled) computer-generated phase plane (in increments of 0.25 from 0 to 2) that the population of both rabbits and foxes decreases until we reach 50 foxes, and then the rabbit population starts to increase followed by the fox population until the fox population recovers to 50. After this, the rabbit population decreases again until we return to the starting populations, and the cycle repeats.

3 a The formulae from the Euler method are

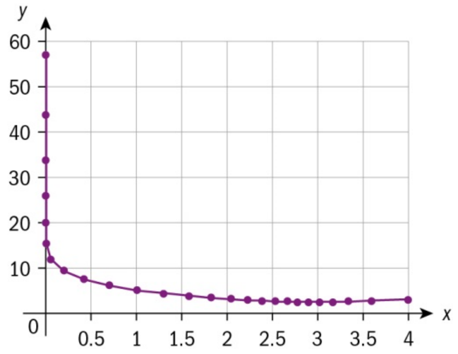
$x_{n+1} = x_n(1.2 - 0.1y_n), y_{n+1} = y_n(1.3 - 0.1x_n)$

and

$x_0 = 4, y_0 = 3$

$x_n < 0.01$ or $y_n < 0.01$ is extinction of X or Y respectively.

Using a spreadsheet gives $x_{20} < 0.01$ and so X goes extinct, while the population of Y continues to climb.



b **i** $\frac{dy}{dt} = 3y$

ii $y = Ae^{3t}$.

c When $x \to 0$, then $\frac{dx}{dt} \to 0$ and so $\frac{dy}{dx} \to \infty$, so the y-axis is an asymptote.

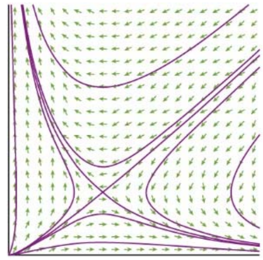
When $y = 0, \frac{dy}{dt} = 0$ and so $\frac{dy}{dx} = 0$, so the x-axis is an asymptote.

d **i** $\frac{dy}{dt} = (3 - x)y < 0$ so y decreases.

ii $\frac{dx}{dt} = (2 - y)x < 0$ so x decreases.

e (0,0) and (3,2)

f



Exercise 12J

1 **a** Let $y = \frac{dx}{dt}$, then $\frac{dy}{dt} = 3y - 2x$

Thus, $x = y$ and $y = 3y - 2x$

We get $M = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}$

$|M - \lambda I| = (\lambda - 1)(\lambda - 2)$

Eigenvectors for $\lambda = 1$ and $\lambda = 2$ are $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

$x = Ae^{t}\begin{pmatrix}1\\1\end{pmatrix} + Be^{2t}\begin{pmatrix}1\\2\end{pmatrix}$

When $t = 0, x = 2$ and $\frac{dx}{dt} = 4$ so $\begin{pmatrix}2\\4\end{pmatrix} = \begin{pmatrix}A+B\\A+2B\end{pmatrix} \Rightarrow A = 0, B = 2$

Thus, $x = 2e^{2t}$

b Let $y = \frac{dx}{dt}$ then $\frac{dy}{dt} = 3y + 4x$

Thus, $x = 0x + 1y$ and $y = 4x + 3y$

We get $M = \begin{pmatrix}0 & 1\\4 & 3\end{pmatrix}$

$|M - \lambda I| = (\lambda - 4)(\lambda + 1)$

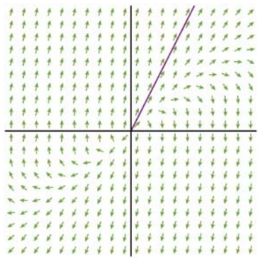
Eigenvectors for $\lambda = -1$ and $\lambda = 4$ are $\begin{pmatrix}1\\-1\end{pmatrix}$ and $\begin{pmatrix}1\\4\end{pmatrix}$.

$x = Ae^{-t}\begin{pmatrix}1\\-1\end{pmatrix} + Be^{4t}\begin{pmatrix}1\\4\end{pmatrix}$

When $t = 0, x = 0$ and $\frac{dx}{dt} = 5$ so $\begin{pmatrix}0\\5\end{pmatrix} = \begin{pmatrix}A+B\\-A+4B\end{pmatrix} \Rightarrow A = -1, B = 1$.

Thus, $x = e^{4t} - e^{-t}$.

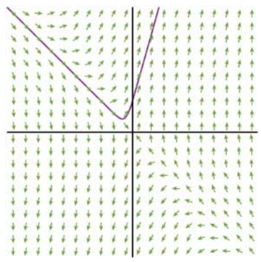
c i In part **a**, $x = 2e^{2t}$ and $y = \frac{dx}{dt} = 4e^{4t}$ hence both x and $\frac{dx}{dt}$ (its velocity) are increasing exponentially, and the value of the velocity (y) is twice the value of x as indicated on the phase diagram below.



ii In part **b**, as $t \to \infty, x \to e^{4t}$ and $\frac{dx}{dt} \to 4e^{4t}$. Hence the relation between y and x for large values of t is $y = 4x$.

as $t \to -\infty, x \to -e^{-t}$ and $\frac{dx}{dt} \to e^{-t}$. Hence the relation between y and x for large values negative of t is $y = -x$

The phase plane is given below.



2 **a** Let $y = x$, then $y = -2x - 3y + 6t + 4$.

b The formulae for the Euler method are $t_{n+1} = t_n + 0.1, x_{n+1} = x_n + 0.1y_n$ and $y_{n+1} = 0.1(7y_n - 2x_n + 6t_n + 4)$.

$x = 6.77$

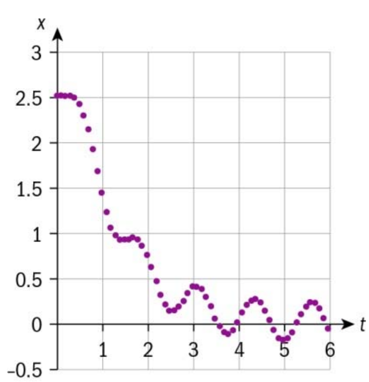
c $x = 5.525$ when $t = 1.6$

d Using the Euler method, for $t > 11$, $\frac{dx}{dt} = 3$. Then $\frac{d^2x}{dt^2} = 0$ and from $y = -2x - 3y + 6t + 4$,

then $x = 3t - \frac{5}{2}$.

3 **a** $\frac{dx}{dt} = y, \frac{dy}{dt} = -2x - 3y + 5\cos(5t)$

b The initial conditions are $x = 2.5$ and $y = 0$ when $t = 0$. Euler's method has $t_{n+1} = t_n + 0.1, x_{n+1} = x_n + 0.1y_n$ and $y_{n+1} = -0.2x_n + 0.7y_n + 0.5\cos(5t_n)$.



We observe that x decays in bursts and has approximate oscillatory behaviour around 0.

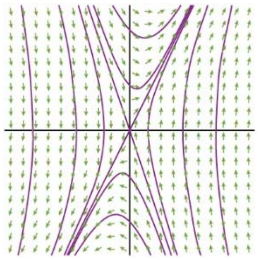
Chapter review

1 **a** $M = \begin{pmatrix} 0 & 1 \\ 6 & -1 \end{pmatrix}$

$|M - \lambda I| = \begin{vmatrix} -\lambda & 1 \\ 6 & -1-\lambda \end{vmatrix} = (\lambda - 2)(\lambda + 3)$, so $\lambda_1 = 2, \lambda_2 = -3$.

Two corresponding eigenvectors are $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -3 \end{pmatrix}$. Thus $x = Ae^{-3t}\begin{pmatrix} 1 \\ -3 \end{pmatrix} + Be^{2t}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

b There is exactly one asymptote, $y = 2x$; $y = -3x$ is not an asymptote.



2 **a** $y = \dot{x}$ and $\dot{y} = -0.4x - e^{0.2t}$

Euler's method gives $t_{n+1} = t_n + 0.1, x_{n+1} = x_n + 0.1y_n$ and $y_{n+1} = y_n + 0.1\left(-0.4x_n - e^{0.2t_n}\right)$ with initial conditions $t_0 = 0, x_0 = 3$ and $y_0 = 0$.

The displacement is first 1.0 cm to 2 significant figures when $t = 1.4$ s.

At this time $x = 1.009$

n	t_n	x_n	y_n
0	0	3	0
1	0.1	3.000	-0.220
2	0.2	2.978	-0.442
3	0.3	2.934	-0.665
4	0.4	2.867	-0.889
5	0.5	2.778	-1.112
6	0.6	2.667	-1.333
7	0.7	2.534	-1.553
8	0.8	2.379	-1.769
9	0.9	2.202	-1.982
10	1.0	2.003	-2.190
11	1.1	1.785	-2.392
12	1.2	1.545	-2.588
13	1.3	1.287	-2.777
14	1.4	1.009	-2.958
15	1.5	0.713	-3.131

b The speed is 2.96 cm s^{-1}.

3 **a** $r = \begin{pmatrix} \frac{4}{3}t^{\frac{3}{2}} + t + c_1 \\ t^2 + c_2 \end{pmatrix}$ by integrating v

When $t = 0, r = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ so $c_1 = c_2 = 0$ and $r = \begin{pmatrix} \frac{4}{3}t^{\frac{3}{2}} + t \\ t^2 \end{pmatrix}$

b When $t = 1, r = \begin{pmatrix} \frac{7}{3} \\ 1 \end{pmatrix}$ and the distance from the origin is $|r| = \sqrt{\left(\frac{7}{3}\right)^2 + 1^2} = \frac{\sqrt{58}}{3} = 2.54$.

4 a If $\frac{dZ}{dt} = 0$, then $Z = 0$ or $L = 2$.

If $Z = 0$, then $\frac{dL}{dt} = 0$ gives $2L - 2L^2 = 0$, so $L = 0$ or $L = 1$.

If $L = 2$, $\frac{dL}{dt} = 0$ gives $4 + 2Z - 8 = 0$, so $Z = 2$.

The equilibrium points are $(L, Z) = (0,0), (1,0)$ or $(2,2)$.

b The Euler method gives $L_{n+1} = L_n + 0.1\left(2L_n + L_nZ_n - 2L_n^2\right)$ and $Z_{n+1} = Z_n + 0.1\left(2Z_n - L_nZ_n\right)$.

i Here $L_0 = 3, Z_0 = 4$ and $L_{20} = 2.021$ and $Z_{20} = 2.023$ giving 20 lions and 2023 zebra.

ii Here $L_0 = 5, Z_0 = 1$ and $L_{20} = 1.764$ and $Z_{20} = 1.670$ giving 18 lions and 1670 zebra.

c The solutions in part **b** are converging to the equilibrium point (2,2), or 20 lions and 2000 zebra. The population will eventually become stable in the absence of any external factors.

5 a $AB = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$ m, $W = F \cdot AB = \begin{pmatrix} 5 \\ 6 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ 4 \end{pmatrix} = 49$ J.

b If $\theta = 0°$ is the angle between the force and the direction of the motion then $W = F \cdot AB = |F| \times |AB| \cos\theta = |F| \times |AB| \cos 0 = |F| \times |AB|$.

c The distance is $\sqrt{40^2 + 55^2 + 3^2} = 68.1$ m and $|F| = 100\,000$ N.

By part **b**, the work done is $100000 \times 68.1 = 6810\,000$ J $= 6810$ kJ.

6 a The resultant force is $F = \begin{pmatrix} 62 \\ 11 \\ 21 \end{pmatrix} + \begin{pmatrix} 62 \\ -11 \\ 21 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 330 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -294 \end{pmatrix} + \begin{pmatrix} 17 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 141 \\ 0 \\ 78 \end{pmatrix}$ kN.

b $a = \frac{1}{m}F = \frac{1}{30000}\begin{pmatrix} 141000 \\ 0 \\ 78000 \end{pmatrix} = \begin{pmatrix} 4.7 \\ 0 \\ 2.6 \end{pmatrix}$ m s^{-2}.

c i By integrating, $v = \begin{pmatrix} 4.7t + c_1 \\ c_2 \\ 2.6t + c_3 \end{pmatrix}$ m s^{-1}. Initial conditions give $c_1 = 55, c_2 = c_3 = 0$.

Thus, $v = \begin{pmatrix} 4.7t + 55 \\ 0 \\ 2.6t \end{pmatrix}$ m s^{-1}.

Integrating again $r = \begin{pmatrix} 2.35t^2 + 55t + c_4 \\ c_5 \\ 1.3t^2 + c_6 \end{pmatrix}$ m. Initial conditions give $c_4 = c_5 = c_6 = 0$ so that $r = \begin{pmatrix} 2.35t^2 + 55t \\ 0 \\ 1.3t^2 \end{pmatrix}$ m.

ii When $t = 60s, r = \begin{pmatrix} 11760 \\ 0 \\ 4680 \end{pmatrix}$ m.

7 a The radius r satisfies $\sin 30 = \frac{r}{40}$, so r = 20 cm.

b Clearly, we have r = 0.2 m. A revolution occurs in 1s, so that $b = \frac{2\pi}{1} = 2\pi$ and

$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0.2\cos 2\pi t \\ 0.2\sin 2\pi t \end{pmatrix}$ m.

c The velocity vector is $x = \begin{pmatrix} -0.4\pi\sin 2\pi t \\ 0.4\pi\cos 2\pi t \end{pmatrix}$ m s^{-1}.

d If the ball is released at time t, the initial speed upon release is

$|x| = \left|\begin{pmatrix} -0.4\pi\sin 2\pi t \\ 0.4\pi\cos 2\pi t \end{pmatrix}\right| = 0.4\pi \left|\begin{pmatrix} -\sin 2\pi t \\ \cos 2\pi t \end{pmatrix}\right| = 0.4\pi = 1.26$ m s^{-1}.

8 a We use a coordinate system so that $A = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $B = \begin{pmatrix} 17 \\ 0 \end{pmatrix}$. $r_A = \begin{pmatrix} 5t \\ 3t - t^2 \end{pmatrix}$ and $r_B = \begin{pmatrix} 17-4t \\ 5t - t^2 \end{pmatrix}$

b The vector is $r_A - r_B = \begin{pmatrix} 9t - 17 \\ -2t \end{pmatrix}$.

c The distance between the beanbags is

$|r_A - r_B| = \left|\begin{pmatrix} 9t - 17 \\ -2t \end{pmatrix}\right| = \sqrt{(9t-17)^2 + (-2t)^2} = \sqrt{85t^2 - 306t + 289}$

This is minimised when $f(t) = 85t^2 - 306t + 289$ is a minimum.

From the GDC the minimum distance is 3.69 units.

d We have $r_A = \begin{pmatrix} at + c_1 \\ bt + c_2 \end{pmatrix}$ now and when $t = 0$, $r_A = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ so $c_1 = c_2 = 0$ and $r_A = \begin{pmatrix} at \\ bt \end{pmatrix}$.

When $t = 1, r_A = r_B$ so that $\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 17-4 \\ 5-1 \end{pmatrix}$ and so $a = 13, b = 4$

e The angle is $\tan^{-1}\left(\frac{4}{13}\right) = 17.1$.

9 Let $\propto$ be the angle of projection.

$a = \begin{pmatrix} 0 \\ -g \end{pmatrix}$ m s^{-2}, so $v = \begin{pmatrix} c_1 \\ -gt + c_2 \end{pmatrix}$ m s^{-1}

The initial velocity is $\begin{pmatrix} 25\cos\alpha \\ 25\sin\alpha \end{pmatrix}$ m s^{-1}, so $v = \begin{pmatrix} 25\cos\alpha \\ -gt + 25\sin\alpha \end{pmatrix}$ m s^{-1}.

When t = 3.5 s, the angle made with the horizontal is 45°.

Therefore, the horizontal and vertical components must be equal and opposite, so

$25\cos\alpha = 3.5g - 25\sin\alpha$

solve to get $\alpha = 31.2°$ or $\alpha = 58.8°$

Exam-style questions

10 a i Integrating, $v(t) = \begin{pmatrix} \frac{t^2}{2} + c_1 \\ 4t + c_2 \end{pmatrix}$. (2)

The object starts at rest, so $v(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and thus, $c_1 = c_2 = 0$. (1)

Thus, $v(t) = \begin{pmatrix} \frac{t^2}{2} \\ 4t \end{pmatrix}$. (1)

ii Integrating, $s(t) = \begin{pmatrix} \frac{t^3}{6} + c_3 \\ 2t^2 + c_4 \end{pmatrix}$. (2)

The object starts at the origin $s(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, so $c_3 = c_4 = 0$. (1)

Thus, $s(t) = \begin{pmatrix} \frac{t^3}{6} \\ 2t^2 \end{pmatrix}$. (1)

b i $v(6) = \begin{pmatrix} 18 \\ 24 \end{pmatrix}$

ii $s(6) = \begin{pmatrix} 36 \\ 72 \end{pmatrix}$ (2)

11 a The scalar component of the force in the direction of NO is

$\frac{|F \cdot NO|}{|NO|} = \frac{|(-12i - 24j)\cdot(-3i - 4j)|}{|-3i - 4j|} = \frac{|36 + 96|}{\sqrt{3^2 + 4^2}} = \frac{132}{5} = 26.4$ N. (3)

b $F \times NO = -24k$

The scalar component of the force perpendicular to NO is $\frac{|F \times NO|}{|NO|} = \frac{24}{5} = 4.8$ N

(3)

c A unit vector in the direction of NO is $\frac{-3i - 4j}{|-3i - 4j|} = -\frac{3}{5}i - \frac{4}{5}j$

The component of F in the direction of NO is thus $26.4\left(-\frac{3}{5}i - \frac{4}{5}j\right) = -15.84i - 21.12j$ N.

(3)

12 a **i** We have $X = Ae^{5t} + Be^{-2t}$, $x = 5Ae^{5t} - 2Be^{-2t}$ and

.

Thus, $x - 3x - 10x = 25Ae^{5t} + 4Be^{-2t} - 15Ae^{5t} + 6Be^{-2t} - 10Ae^{5t} - 10Be^{-2t} = 0$. (3)

ii $x(0) = 0 \Rightarrow A + B = 0$ and $\frac{dx}{dt} = 14$ when $t = 0$ yields $5A - 2B = 14$.

We get $A = 2, B = -2$ and so $x = 2e^{5t} - 2e^{-2t}$. (3)

b **i** We have $X = A + Be^{-3t}$, $x = -3Be^{-3t}$ and $x = 9Be^{-3t}$.

Thus, $x + 3x = 9Be^{-3t} - 9Be^{-3t} = 0$. (3)

ii $x(0) = 0 \Rightarrow A + B = 0$ and $\frac{dx}{dt} = 9$ when $t = 0$ yields $9 = -3B$.

We get $A = 3, B = -3$ and so $x = 3 - 3e^{-3t}$. (3)

iii $x(t) \to 3$ as $t \to \infty$ since $e^{-3t} \to 0$ as $t \to \infty$. (1)

13 a $M = \begin{pmatrix} 2 & -3 \\ 0 & -1 \end{pmatrix}$

$|M - \lambda I| = \begin{vmatrix} 2-\lambda & -3 \\ 0 & -1-\lambda \end{vmatrix} = (\lambda - 2)(\lambda + 1)$, so $\lambda_1 = 2, \lambda_2 = -1$. (3)

$\begin{pmatrix} 2 & -3 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = 2\begin{pmatrix} p \\ q \end{pmatrix}$ gives $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ as an eigenvector (1)

$\begin{pmatrix} 2 & -3 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = -1\begin{pmatrix} p \\ q \end{pmatrix}$ gives $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ as an eigenvector (1)

Thus, $x = Ae^{-t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + Be^{2t}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ (1)

When $t = 0, x = \begin{pmatrix} 6 \\ 4 \end{pmatrix}$. (2)

Therefore, $\begin{pmatrix} 6 \\ 4 \end{pmatrix} = \begin{pmatrix} A+B \\ A \end{pmatrix}$ so $A = 4, B = 2$. We get $x = 4e^{-t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + 2e^{2t}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. (2)

b When 1000 years pass, $t = 1$ and $x = \begin{pmatrix} 4e^{-1} + 2e^{2} \\ 4e^{-1} \end{pmatrix} = \begin{pmatrix} 16 \\ 1.5 \end{pmatrix}$ to 2 s.f., giving 160 000 *Homo sapiens* and 15 000 Neanderthals. (2)

c **i** $x = 4e^{-t} + 2e^{2t} \to \infty$ as $t \to \infty$

ii $y = 4e^{-t} \to 0$ as $t \to \infty$. Dies out. (2)

14 a $a = \dot{v} = \begin{pmatrix} -105\pi(-3\pi \sin 3\pi t) \\ -105\pi(3\pi \cos 3\pi t) \end{pmatrix} = \begin{pmatrix} 315\pi^2 \sin 3\pi t \\ -315\pi^2 \cos 3\pi t \end{pmatrix}$ cm s^{-2}. (3)

b Integrating v, $s=\begin{pmatrix}-35\sin 3\pi t+c_1\\ 35\cos 3\pi t+c_2\end{pmatrix}$ cm. (2)

When $t=0, s=\begin{pmatrix}0\\75\end{pmatrix}$ cm, so $c_1=0$ and $c_2=40$ cm. (2)

Thus, $s=\begin{pmatrix}-35\sin 3\pi t\\ 35\cos 3\pi t+40\end{pmatrix}$ cm. (2)

c The condition given is $-315\pi^2\cos 3\pi t=0\Rightarrow\cos 3\pi t=0$. (1)

Within a single cycle of the wheel solutions are $t=\frac{1}{6},\frac{1}{2}$

Substituting into the equation for displacement

the possible positions for the logo are $\begin{pmatrix}-35\\40\end{pmatrix}$ cm and $\begin{pmatrix}35\\40\end{pmatrix}$ cm. (2)

d Period $=\frac{2\pi}{3\pi}=\frac{2}{3}$ s, so in 1 minute there are $60\times\frac{3}{2}=90$ revolutions. (2)

15 a $M=\begin{pmatrix}2&1\\1&2\end{pmatrix}$

$|M-\lambda I|=\begin{vmatrix}2-\lambda&1\\1&2-\lambda\end{vmatrix}=(\lambda-3)(\lambda-1)$, so $x=Ae^{t}p_1+Be^{3t}p_2$. (2)

Both eigenvalues are positive and the solutions move away from 0 here. (1)

b $M=\begin{pmatrix}-2&1\\1&-2\end{pmatrix}$

$|M-\lambda I|=\begin{vmatrix}-2-\lambda&1\\1&-2-\lambda\end{vmatrix}=(\lambda+1)(\lambda+3)$, so $x=Ae^{-t}p_1+Be^{-3t}p_2$. (2)

Both eigenvalues are negative and the solutions move towards 0 here. (1)

c $M=\begin{pmatrix}2&1\\-1&2\end{pmatrix}$

$|M-\lambda I|=\begin{vmatrix}2-\lambda&1\\-1&2-\lambda\end{vmatrix}=(\lambda-2)^2+1$, so the eigenvalues are $2\pm i$. (2)

The real part is positive and the solutions spiral away from the origin. (1)

d $M=\begin{pmatrix}-2&-1\\1&-2\end{pmatrix}$

$|M-\lambda I|=\begin{vmatrix}-2-\lambda&-1\\1&-2-\lambda\end{vmatrix}=(\lambda+2)^2+1$, so the eigenvalues are $-2\pm i$. (2)

The real part is negative and the solutions spiral towards the origin. (1)

16 a $x'(t)=x-xy=0,\ y'(t)=xy-2y\Rightarrow x=0, y=0$ or $x=2, y=1$ (3)

The equilibrium points are (0,0) and (2,1), or 2000 lemmings and 100 owls.

b Euler's method gives $x_{n+1} = x_n(1.25 - 0.25y_n)$ and $y_{n+1} = y_n(0.5 + 0.25x_n)$ with initial conditions $x_0 = 1$ and $y_0 = 0.5$. (1)

Tabulating the results below:

t	x_n	y_n
0	1	0.5
0.25	1.1250	0.3750
0.50	1.3008	0.2930
0.75	1.5307	0.2418
1.00	1.8209	0.2134
1.25	2.1789	0.2038
1.50	2.6126	0.2130
1.75	3.1267	0.2456
2.00	3.7164	0.3147

(4)

We get approximately 3716 lemmings and 31 owls. (2)

c It is to 3 decimal places for x and seems a reasonable level of accuracy. However, Euler's method with step size t=0.25 does lead to an unphysical negative value of x_{15}, and is somewhat suspect. (2)

17 Let ϑ be the angle between a and b, and n a unit vector perpendicular to both a and b, so that $|n| = 1$.

We have $a \cdot b = |a||b|\cos\theta$ and $a \times b = |a||b|\sin\theta n$.

Therefore $\left(\frac{a \cdot b}{|b|}\right)^2 + \left(\frac{|a \times b|}{|b|}\right)^2 = \left(\frac{|a||b|\cos\theta}{|b|}\right)^2 + \left(\frac{\big||a||b|\sin\theta n\big|}{|b|}\right)^2$ (2)

$= (|a|\cos\theta)^2 + (|a|\sin\theta|n|)^2 = |a|^2(\cos^2\theta + \sin^2\theta)$ (2)

$= |a|^2$ (1)

18 We have $\frac{dx}{dt} = 0x + 1y, \frac{dy}{dt} = -cx - by$. In the usual notation $M = \begin{pmatrix} 0 & 1 \\ -c & -b \end{pmatrix}$. (2)

Now $|M - \lambda I| = \begin{vmatrix} -\lambda & 1 \\ -c & -b-\lambda \end{vmatrix} = \lambda^2 + b\lambda + c$. (2)

So the eigenvalues are λ_1 and λ_2. (1)

Two eigenvectors that correspond to each eigenvalue are $\begin{pmatrix} 1 \\ \lambda_1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ \lambda_2 \end{pmatrix}$ respectively, since λ_1 and λ_2 are both roots of the quadratic $\lambda^2 + b\lambda + c = 0$. (4)

Then $x = Ae^{\lambda_1 t}\begin{pmatrix} 1 \\ \lambda_1 \end{pmatrix} + Be^{\lambda_2 t}\begin{pmatrix} 1 \\ \lambda_2 \end{pmatrix}$, so that $x = Ae^{\lambda_1 t} + Be^{\lambda_2 t}$ as required. (2)

13 Representing multiple outcomes: random variables and probability distributions

Skills check

1 Mean $= \dfrac{1\times 9+2\times 7+3\times 3+6\times 2+11\times 1}{9+7+3+2+1} = \dfrac{55}{22} = 2.5$

2 Using technology, the solution is $x \approx 1.39$

Exercise 13A

1 **a** Not a discrete distribution as the sum of the probabilities is greater than 1

b Not a discrete distribution as one of the probabilities is negative

c This is a discrete probability distribution

2 **a** $P(A=12) = 1-(0.5+0.05+0.04+0.1+0.2) = 0.11$

b $P(8 < A \le 10) = P(A=9)+P(A=10) = 0.04+0.1 = 0.14$

c $P(A \le 9) = P(A=5)+P(A=8)+P(A=9) = 0.5+0.05+0.04 = 0.59$

d $P(A \ge 10) = P(A=10)+P(A=11)+P(A=12) = 0.1+0.2+0.11 = 0.41$

e $P(A>8 \mid A \le 11) = \dfrac{P(9 \le A \le 11)}{P(A \le 11)} = \dfrac{0.04+0.1+0.2}{0.5+0.05+0.04+0.1+0.2} = 0.38$

f $E(A) = 5\times 0.5+8\times 0.05+9\times 0.04+10\times 0.1+11\times 0.2+12\times 0.11 = 7.78$

3 **a** $P(B=1) = 0.0001$, $P(B=2) = 0.9999\times 0.0001 = 0.00009999$,
$P(B=3) = 0.9999\times 0.9999\times 0.0001 = 0.00009998$

b To win on your b-th crisp packet, you need to have had $(b-1)$ losses and then a win on the b-th try, so $P(B=b) = P(\text{lose})^{b-1}\times P(\text{win}) = 0.0001\times(0.9999)^{b-1}$

c The domain is the set of positive integers, 1, 2, … That is, $b \in \mathbb{Z}^+$.

d $P(B \le 10) = P(B=1)+\cdots+P(B=10) = 0.00099955$

4 **a**

c	1	2	3	4	5
$P(C=c)$	0.07	0.02	0.17	0.46	0.28

b $E(C) = 1\times 0.07+2\times 0.02+3\times 0.17+4\times 0.46+5\times 0.28 = 3.86$

5 a

	1	**2**	**3**	**4**
1	1	2	3	4
2	2	4	6	8
3	3	6	9	12
4	4	8	12	16

d	1	2	3	4	6	8	9	12	16
$f(d)$	1	2	2	3	2	2	1	2	1
$P(D = d)$	$\frac{1}{16} =$ 0.0625	$\frac{1}{8} =$ 0.125	$\frac{1}{8} =$ 0.125	$\frac{3}{16} =$ 0.1875	$\frac{1}{8} =$ 0.125	$\frac{1}{8} =$ 0.125	$\frac{1}{16} =$ 0.0625	$\frac{1}{8} =$ 0.125	$\frac{1}{16} =$ 0.0625

b $P(D \text{ is a square number} \mid D < 8) = \dfrac{P(D \text{ is a square number} \cap D < 8)}{P(D < 8)}$

$$= \frac{P(D=1)+P(D=4)}{P(D=1)+\cdots+P(D=6)} = \frac{0.0625+0.1875}{0.0625+\cdots+0.125} = \frac{2}{5} = 0.4$$

6

No of heads	0	1	2	3
Outcomes	TTT	TTH, THT, HTT	THH, HTH, HHT	HHH
m	2	y	5	15
$P(M = m)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

a $E(M) = 2\times\frac{1}{8} + y\times\frac{3}{8} + 5\times\frac{3}{8} + 15\times\frac{13}{8} = 4 + \frac{3}{8}y$

b To be fair, we want $E(M) = \$7$, so $7 = 4 + \frac{3}{8}y \Rightarrow y = 8$

7 a

f	0	1	2
$P(F = f)$	$\frac{4}{7}\times\frac{3}{6} = \frac{2}{7}$	$\frac{4}{7}\times\frac{3}{6} + \frac{3}{7}\times\frac{4}{6} = \frac{4}{7}$	$\frac{3}{7}\times\frac{2}{6} = \frac{1}{7}$

b $E(F) = 0\times\frac{2}{7} + 1\times\frac{4}{7} + 2\times\frac{1}{7} = \frac{6}{7}$

8 a i $P(G = 1) = \frac{1}{5} = \frac{9}{45}$, $P(G = 2) = \frac{8}{10}\times\frac{2}{9} = \frac{8}{45}$, $P(G = 3) = \frac{8}{10}\times\frac{7}{9}\times\frac{2}{8} = \frac{7}{45}$, …

Thus, $P(G=g)=\dfrac{10-g}{45}$ for $g=1,\ldots,9$ (it is not possible to pick the first black one on the 10th go).

ii $E(G)=1\times\dfrac{1}{5}+2\times\dfrac{8}{45}+\cdots+8\times\dfrac{2}{45}+9\times\dfrac{1}{45}=\dfrac{11}{3}$

b i $P(G=1)=\dfrac{1}{5}$, $P(G=2)=\dfrac{8}{10}\times\dfrac{2}{10}=\dfrac{4}{25}$, $P(G=3)=\dfrac{8}{10}\times\dfrac{8}{10}\times\dfrac{2}{10}=\dfrac{16}{125}$, …,

$P(G=g)=\left(\dfrac{8}{10}\right)^{g-1}\times\dfrac{2}{10}$ for any $g=1,2,\ldots$

ii $E(G)=1\times\dfrac{1}{5}+2\times\dfrac{4}{25}+\cdots+g\times\left(\dfrac{8}{10}\right)^{g-1}\times\dfrac{2}{10}+\cdots=5$

c For **a**: This is an arithmetic series: $\sum_{g=1}^{9}P(G=g)=\sum_{g=1}^{9}\left(\dfrac{10-g}{45}\right)=\dfrac{9}{2}\left(\dfrac{9}{45}+\dfrac{1}{45}\right)=1$

For **b**: This is a geometric series: $\sum_{g=1}^{\infty}P(G=g)=\sum_{g=1}^{\infty}\left(\left(\dfrac{8}{10}\right)^{g-1}\times\dfrac{2}{10}\right)=\dfrac{1}{5}\left(\dfrac{1}{1-\frac{4}{5}}\right)=1$

9 a $P(B\geq x+y\mid B\geq x)=\dfrac{P(B\geq x+y)}{P(B\geq x)}=\dfrac{1-P(B\leq x+y-1)}{1-P(B\leq x-1)}$

$=\dfrac{1-\left(1-(1-p)^{x+y-1}\right)}{1-\left(1-(1-p)^{x-1}\right)}=(1-p)^{y}=1-F(y)=P(B\geq y+1)$

b This means that the process has no "memory". The probability of finding a golden ticket after at least 15 trials given that no golden ticket was found after at least 10 has the same probability as starting over.

Exercise 13B

1 a $P(X=4)=0.0535$

b $P(X\leq 4)=0.991$

c $P(1\leq X<4)=P(X=1)+P(X=2)+P(X=3)=0.809$

An alternative is to find $P(X\leq 3)-P(X\leq 0)$

d $P(X\geq 2)=0.558$

On some calculators this needs to be calculated as $1-P(X\leq 1)$

e $P(X\leq 4\mid X\geq 2)=\dfrac{P(X\leq 4\cap X\geq 2)}{P(X\geq 2)}=\dfrac{P(2\leq X\leq 4)}{P(X\geq 2)}$

$=0.983$

Depending on the GDC the numerator in the last expression can be calculated directly or as $P(X=2)+P(X=3)+P(X=4)$ or as $P(X\leq 4)-P(X\leq 2)$

f $X\leq 4$ and $X\geq 2$ are not independent because
$P(X\leq 4\mid X\geq 2)=0.9833\neq 0.9907=P(X\leq 4)$

2 $P(\text{prime}) = \dfrac{n(\{2,3,5,7\})}{n(\{1,2,3,4,5,6,7,8\})} = \dfrac{1}{2}$,

$P(\text{at least three primes}) = P(\text{number of primes} \geq 3) = 0.773$

On some GDCs this can be calculated as
$P(\text{at least three primes}) = 1 - P(\text{number of primes} \leq 2) = 0.773$

3 a $E(Y) = 5.2 = n \times 0.4 \Rightarrow n = 13 \Rightarrow \text{Var}(Y) = 13 \times 0.4 \times 0.6 = 3.12$

b $\text{Var}(Z) = 1.44 = 9 \times p \times (1-p) \Rightarrow 9p^2 - 9p + \dfrac{36}{25} = 0 \Rightarrow p = \dfrac{1}{5}$ or $p = \dfrac{4}{5}$

4 a Distribution is $B(10, 0.1)$

$P(X \geq 5)\left(= 1 - P(X \leq 4)\right) = 0.00163$

b Probability of no points in a single game = $P(X = 0) = 0.3486...$

Let Y be the number of times in 6 games that no points are scored.

Distribution of Y is $B(6, 0.3486...)$

$$\begin{aligned} P(\text{no points in at least 2 games}) &= P(Y \geq 2) = 1 - P(Y \leq 1) \\ &= 1 - 0.32156... \\ &\approx 0.678 \end{aligned}$$

5 a $A \sim B(25, 0.2)$

b $P(A \leq 5) = 0.617$

c $P(A \geq 7) = 1 - P(A \leq 6) = 0.220$

d $P(A \leq 3) = 0.234$

e $E(A) = 25 \times 0.2 = 5$ so it is expected that he will guess 5 questions correctly

f $P(A > 5) = 1 - P(A \leq 5) = 1 - 0.617 = 0.383$

g Expected points $= 4 \times 5 + (-1) \times 20 = 0$

h $P(\text{at least 2 get 7 or more}) = 0.212$

6 a $T \sim B(538, 0.91)$ assuming that the arrival of a passengers is independent of the arrival of any other passenger.

b $P(T = 538) = 0.91^{538} = 9.21 \times 10^{-23}$; it is exceedingly unlikely that all passengers will turn up on time to take the flight

c $P(T \geq 510) = 0.000672$; it is very likely that there will be at least 28 empty seats on the plane

d Consider the distribution $T_n \sim B(n, 0.91)$ where S is the number of passengers who turn up when n tickets are sold.

We need to find the smallest value of n such that $P(T_n \geq 510) \geq 0.1$

This can be entered into the function menu of the GCC as $1-Binomcdf(x,0.91,509)$ (or similar depending on the names given to the cumulative distribution function on the GDC). The Table can then be used to find the smallest value of x (or n) for which this is greater than 0.1

$n=550$, $P(T\geq 510)=0.0870$. For $n=551$, $P(T\geq 510)=0.112$.

e $E(T)=538=n\times 0.91\Rightarrow n=591$

f $P(T=538)=0.0573$
$P(T>538)=P(X\geq 539)=1-P(X\leq 538)=0.468$
It is now much more likely that the plane will be full, and it is also very likely that the plane will be overbooked.

7 $\text{Var}(X)=np(1-p)$. Differentiating to find the max, $n(1-2p)=0\Rightarrow p=\frac{1}{2}$

8 **a** Let X be the number of times 1 is scored when the 3 spinners are spun hence $X\sim B(3,0.2)$

Expected frequencies:
0 1s thrown: $200\times P(X=0)=102.4$
1 1s thrown: $200\times P(X=1)=76.8$
2 1s thrown: $200\times P(X=2)=19.2$
3 1s thrown: $200\times P(X=3)=1.6$

b There are large differences between the expected numbers and the actual recorded numbers, indicating that the dice may not be fair.

Exercise 13C

1 **a** $P(A=2)=0.0390$

b $P(A<6)=P(A\leq 5)=0.414$s

c $P(A\geq 7\mid A>5)=\frac{P(A\geq 7\cap A>5)}{P(A>5)}=\frac{P(A\geq 7)}{P(A>5)}=\frac{1-P(A\leq 6)}{1-P(A\leq 5)}=0.727$

d $P(A\leq 4)=0.259$

e $P(A>8\mid A\geq 3)=\frac{P(A>8\cap A\geq 3)}{P(A\geq 3)}=\frac{P(A>8)}{P(A\geq 3)}=\frac{1-P(A\leq 8)}{1-P(A\leq 2)}=0.184$

2 $\frac{\beta^0 e^{-\beta}}{0!}=0.301\Rightarrow \beta=1.201\Rightarrow \text{Var}(Z)=\beta=1.20$

or solve $Poissonpdf(\beta,0)=0.301$ using the graph or equation solving facility on your GDC

3 $\frac{\alpha^1 e^{-\alpha}}{1!}=0.15\Rightarrow \alpha=0.179$ or 2.99.

or solve $Poissonpdf(\alpha,1)=0.15$ using the graph or equation solving facility on your GDC

For $\alpha=0.179$, $P(Y=1)=0.15$.

For $\alpha=2.99$, $P(Y=1)=0.15$.

The maximum probability will be close to the mean. For $\alpha = 2.99$, $Y = 1$ is before the mean as the probabilities are increasing and for $\alpha = 0.179$, $Y = 1$ is after the mean as they decrease.

4 $S \sim \text{Po}(27.8)$. It is assumed that the seeds are distributed randomly throughout the 10 loaves. The most likely number of seeds per loaf is either 27 or 28. Since $P(S = 27) = 0.07566$ and $P(S = 28) = 0.07512$, then the most likely number of seeds in a loaf is 27.

5 To make this fair, need $E(\text{Consequence}) = 0$ so we need to find x such that

$$0 = -10000 \times P(C \geq 4) + 0 \times P(1 \leq C \leq 3) + x \times P(C = 0)$$
$$= -10000 \times \left(1 - P(C \leq 3)\right) + x \times P(C = 0)$$
$$= -2026.89 + 0.09926x$$
$$\Rightarrow x = £20420$$

6 The mean number of calls is $\dfrac{0\times15 + 1\times30 + 2\times28 + 3\times14 + 4\times7 + 5\times8}{15+30+28+14+7+8} = \dfrac{196}{102} = 1.9216$, so the following table shows the expected frequencies if the data follows a Poisson distribution.

Number of calls	0	1	2
Expected number of hours	$102 \times P(X = 0)$ $= 14.9$	$102 \times P(X = 1)$ $= 28.7$	$102 \times P(X = 2)$ $= 27.6$
Number of calls	3	4	5 or more
Expected number of hours	$102 \times P(X = 3)$ $= 17.7$	$102 \times P(X = 4)$ $= 8.48$	$102 \times P(X = 5)$ $= 3.26$

Indicating that the data is well modelled by a Poisson distribution except for "5 ". This might be because these calls to the helpline are not independent and are due to a problem affecting several computers at the same time.

7 **a** Let $A \sim \text{Po}(4.2 \times 2)$ and $B \sim \text{Po}(1.7 \times 2)$, then

$P(A = 0 \cap B = 0) = P(A = 0) \times P(B = 0) = 7.50 \times 10^{-6}$

b Let $C \sim \text{Po}((4.2 + 1.7) \times 2)$, then $P(C = 0) = 7.50 \times 10^{-6}$

8 **a** Let $A \sim \text{Po}(5 \times 0.597)$, then $P(A = 0) = 0.0505$

b Let $B \sim \text{Po}(10 \times 24 \times 0.597)$, then

$P(B \geq 3) = 1 - P(B \leq 2) = 1$

c Let $C \sim \text{Po}(24 \times 0.597)$, then $P(C < 11) = P(C \leq 10) = 0.1549$

Let X represent the number of days in a week on which there are fewer than 11 breakdowns. Hence $X \sim B(7, 0.1549)$

$P(X \geq 5) = 1 - P(X \leq 4) = 0.00142$

9 **a** For $R \sim \text{Po}(5.1)$, $365 \times P(R = 0) = 2.23$

b $100 \times P(R > 10) = 100 \times (1 - P(R \leq 10)) = 1.56$

Exercise 13D

1 It is known that $P(X>\mu)=P(X<\mu)=0.5$

From investigation 10 we have

If $X \sim N(\mu, \sigma^2)$, then:	
$P(\mu-\sigma \le X \le \mu+\sigma)=$	**0.68**
$P(\mu-2\sigma \le X \le \mu+2\sigma)=$	**0.95**
$P(\mu-3\sigma \le X \le \mu+3\sigma)=$	**0.997**

a $P(T<17.1)=P(T<\mu)=0.5$

b $P(T<14)=P\left(T<\mu-\sigma\right)\approx\dfrac{1-0.68}{2}=0.16$

c $P(T>20.2)=P\left(T>\mu+\sigma\right)\approx\dfrac{1-0.68}{2}=0.16$

d $P(14\le T<23.3)=P\left(\mu-\sigma\le T<\mu+2\sigma\right)$

$=P(T<\mu+2\sigma)-P\left(T<\mu-\sigma\right)\approx 0.95+0.025-\dfrac{1-0.68}{2}=0.82$

e $P(T<7.8)=P\left(T<\mu-3\sigma\right)\approx\dfrac{1-0.997}{2}=0.0015$

f $P(T<23.3\,|\,T>20.2)=\dfrac{P(\mu+\sigma<T<\mu+2\sigma)}{P(T>\mu+\sigma)}$

$=\dfrac{P(T<\mu+2\sigma)-P(T<\mu+\sigma)}{P(T>\mu+\sigma)}\approx\dfrac{0.975-0.84}{0.16}$

$=0.84$

2 **a** $P(Q<4)=0.483$

b $P(Q<3.4)=0.184$

c $P(Q>5)=0.0829$

On some GDCs this is calculated using $P(Q>5)=1-P(Q<5)$

d $P(3.5\le Q<4.5)=0.525$

On some GDCs this is calculated using $P(Q<4.5)-P(Q<3.5)$

e $P(Q<4.9\,|\,Q>2.9)=\dfrac{P(2.9<Q<4.9)}{P(Q>2.9)}=0.887$

3 If $F(a)=P(X<a)=p$ then define $F^{-1}\left(p\right)=a$ This is normally referred to as the inverse normal function on a GDC. You will need to enter the value for *p* as well as the mean and standard deviation.

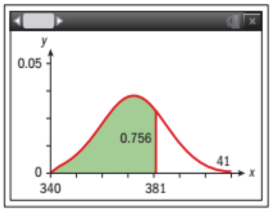
$Q_3=F^{-1}(0.75)=22331.3$

$Q_1 = F^{-1}(0.25) = 21926.7$

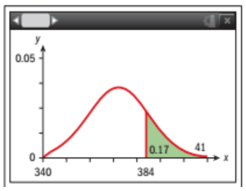
$IQR = Q_3 - Q_1 = 22331.3 - 21926.7 = 405$

4 $P(|S| > k) = 0.57 \Rightarrow P(S < -k) = 0.285$, so $-k = F^{-1}(0.285) \Rightarrow k = 0.767$

5 **a** $P(X \le 381) = 0.756$



b $P(X > t) = 0.17 \Rightarrow P(X < t) = 1 - 0.17 = 0.83 \Rightarrow t = F^{-1}(0.83) = 0.384$



6 $L \sim \mathrm{N}(182, 10^2)$

a $P(L > 190) = 0.212$

b Let X be the number of batteries in the sample that last longer than 190 days. Hence $X \sim B(7, 0.2119...)$

$P(X \le 3) = 0.959$

c $P(L \le 165) = 0.0446$

d $10000 \times 0.04457 = 446$

7 $D \sim \mathrm{N}(16, 5^2)$

a $P(13 < D < 15.3) = 0.170$

b $P(D > x) = 0.13 \Rightarrow P(D < x) = 0.87 \Rightarrow x = F^{-1}(0.87) = 21.6$

c $23109 \times P(D > 14) = 23109 \times 0.6554 = 15146$

Of these 15146 employees who live more than 14 km from work, $0.91 \times 15146 = 13783$ will fail to get to work.

8 $P(M \le m) = 0.99 \Rightarrow m = F^{-1}(0.99) = 34.6$ so 35 minutes to be sure of meeting the target.

9 **a** Route A is on average shorter than route B, but has more variability, so the nurse will have to allow time for a longer journey. Route B takes on average longer but has less variability so the actual times are closer to the average.

b $A \sim \mathrm{N}(42, 8^2)$, $B \sim \mathrm{N}(50, 3^2)$:
$P(A < 45) = 0.6462$
$P(B < 45) = 0.04779$
He should take route A.

c Let C be the number of days on which he arrives on time. $C \sim B(5, 0.6462)$

i $P(C = 5) = 0.6462^5 = 0.113$

ii $P(C \geq 3) = 1 - P(C \leq 2) = 0.759$

iii Let T represent thr event arriving on time. There are three possibilities for the three consecutive days $TTTT'T', T'TTTT'$ and $T'T'TTT$

Each of these have the same probability so
$P(\text{On time on 3 consecutive days}) = 3 \times 0.6462^3 \times 0.3538^2$
$= 0.101$

Exercise 13E

1 **a** $6 = E(3F + 1) = 3E(F) + 1 \Rightarrow E(F) = \dfrac{5}{3}$

b $7 = \text{Var}(3.1 - 2F) = (-2)^2 \times \text{Var}(F) \Rightarrow \text{Var}(F) = \dfrac{7}{4}$

2 Because these have a Poisson distribution, $E(C) = \text{Var}(C)$ and $E(D) = \text{Var}(D)$.

a $E(9C - 4D) = 9E(C) - 4E(D) = 9 \times 3 - 4 \times 6.1 = 2.6$

b $\text{Var}(D + 0.2C) = \text{Var}(D) + 0.2^2 \times \text{Var}(C) = 6.1 + 0.04 \times 3 = 6.22$

3 **a** $E(2U + 9) = 2E(U) + 9 = 2 \times 4.01 + 9 = 17.02$

b $E(4X - 0.1) = 4E(X) - 0.1 = 4 \times 7.81 - 0.1 = 31.14$

c $\text{Var}(0.9W + 10) = 0.9^2 \times \text{Var}(W) = 0.81 \times 3 = 2.43$

d $\text{Var}(7 - 3V) = 3^2 \times \text{Var}(V) = 9 \times 0.4 = 3.6$

e $E(U + 4X - 2W) = E(U) + 4E(X) - 2E(W) = 4.01 + 4 \times 7.81 - 2 \times 7.81 = 9.45$

f $\text{Var}(2V + 0.8U - 0.9X + W) = 2^2 \times \text{Var}(V) + 0.8^2 \times \text{Var}(U) + (-0.9)^2 \times \text{Var}(X) + \text{Var}(W)$
$= 4 \times 0.4 + 0.64 \times 1.2 + 0.81 \times 2.11 + 3 = 7.08$

4 $5.1 = E(aU + b) = aE(U) + b = 4.01a + b$ and $2 = \text{Var}(aV + b) = a^2\text{Var}(V) = 0.4a^2$. Solving these gives $a = 2.24$ and $b = -3.87$

5 $6 = E(aW + bX) = aE(W) + bE(X) = 12.9a + 7.81b$ and
$2 = E(aU + bV) = aE(U) + bE(V) = 4.01a + 2.7b$. Solving these gives $a = 0.165$ and $b = 0.496$

6 $15.1 = E(aW + bX) = aE(W) + bE(X) = 12.9a + 7.81b$ and
$2 = \text{Var}(aU + bV) = a^2\text{Var}(U) + b^2\text{Var}(V) = 1.2a^2 + 0.4b^2$. Solving these gives $a = 1.29$ and $b = -0.191$

7 **a**

t	1	2
$P(T = t)$	$\dfrac{1}{3}$	$\dfrac{2}{3}$

h	3	4	5
$P(H=h)$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$

b **i** $E(T) = 1\times\frac{1}{3} + 2\times\frac{2}{3} = \frac{5}{3}$

ii $E(H) = 3\times\frac{1}{4} + 4\times\frac{1}{4} + 5\times\frac{1}{2} = \frac{17}{4}$

c $E(S) = E(T+H) = E(T) + E(H) = \frac{5}{3} + \frac{17}{4} = \frac{71}{12}$

d

s	4	5	6	7
$P(S=s)$	$\frac{1}{12}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{3}$

8 Let Z_1 and Z_2 be the results shown on dice 1 and 2, respectively.

Game 1:

$$E(K) = E(Z_1 + Z_2) = E(Z_1) + E(Z_2) = \frac{3+4+5+5}{4} + \frac{3+4+5+5}{4} = \frac{17}{2}$$

$$Var(K) = \text{Var}(Z_1 + Z_2) = \text{Var}(Z_1) + \text{Var}(Z_2) = \frac{11}{16} + \frac{11}{16} = \frac{11}{8}$$

Outcomes = {6,7,8,9,10}

Game 2:

$$E(L) = E(2Z_1) = 2E(Z_1) = 2\times\frac{17}{4} = \frac{17}{2}$$

$$Var(L) = \text{Var}(2Z_1) = 4\text{Var}(Z_1) = 4\times\frac{11}{16} = \frac{11}{4}$$

Outcomes = {6,8,10}

The variances and outcomes are different, so the distributions are different too and Zeinab is right.

Exercise 13F

1 $I \sim \text{Po}(0.7)$ and $M \sim \text{Po}(0.6)$, so $I + M = W \sim \text{Po}(0.7+0.6)$

$P(W \geq 1) = 1 - P(W=0) = 0.728$

2 $M \sim \text{N}(85, 10^2)$ and $F \sim \text{N}(60, 7^2)$, so

$M_1 + M_2 + F_1 + F_2 + F_3 = T \sim \text{N}(2\times 85 + 3\times 60, 2\times 10^2 + 3\times 7^2) = N(350, 347)$

$P(T > 375) = 0.0898$

3 $P \sim \text{N}(3\times 300 + 2\times 180 + 2\times 100 + 16, 3\times 2^2 + 2\times 7^2 + 2\times 2^2 + 0.5^2) = N(1476, 118.25)$,

$P(P > 1500) = 0.0137$

4 $E \sim \text{Po}(24\times 1.5 + 2) = \text{Po}(38)$

$P(E > 40) = 1 - P(E \le 40) = 0.334$

5 **a** Let X be the difference in length between the length of one short and one regular together and one long

$$X \sim \mathrm{N}(40+80-120, 2.1^2+3.7^2+4^2) = \mathrm{N}(0, 34.1)$$

$P(X > 0) = 0.5$

b $X \sim \mathrm{N}(3\times 40-120, 3\times 2.1^2+4^2) = \mathrm{N}(0, 29.23)$

$P(X < 0) = 0.5$

6 **a** By the central limit theorem

$E(\bar{A}) = E(A) = 110$ and $\mathrm{Var}(\bar{A}) = \dfrac{\mathrm{Var}(A)}{40} = 15.625$

$E(\bar{B}) = E(B) = 123$ and $\mathrm{Var}(\bar{B}) = \dfrac{\mathrm{Var}(B)}{40} = 1.640$

Thus, $\bar{A}-\bar{B} \sim \mathrm{N}(110-123, 15.625+1.640) = \mathrm{N}(-13, 17.265)$

$P(\bar{A}-\bar{B} > 0) = 0.00878$

b $\bar{A} \sim N(110, 15.625)$, so $P(108 < \bar{A} < 112) = 0.387$

c $\bar{B} \sim N\left(123, \dfrac{8.1^2}{50}\right) = N(123, 1.31)$, so $P(\bar{B} > 125) = 0.0404$.

So the expected number of samples is $150 \times 0.0404 = 6.06$

7 $2C - H \sim \mathrm{N}(2\times 5.1-10, 2^2\times 0.1^2+0.1^2) = \mathrm{N}(0.2, 0.05)$

$P(2C - H < 0) = 0.186$

8 $\bar{A} \sim N\left(172, \dfrac{7^2}{25}\right)$

$P(\bar{A} > 175) = 0.0161$

$\bar{E} \sim N\left(170, \dfrac{7^2}{25}\right)$

$P(\bar{E} > 175) = 0.000178$

9 $E(\bar{C}) = 170$ and $\mathrm{Var}(\bar{A}) = \dfrac{6^2}{n}$

This can be solved by educated guessing or by putting a function similar to following into the GDC and using the table to find the value of n.

$$F = normcdf\left(172, 170, \frac{6}{\sqrt{x}}\right) - normcdf\left(168, 170, \frac{6}{\sqrt{x}}\right)$$

If $n = 24$ then $P(168 < \bar{C} < 172) = 0.898$

If $n = 25$ then $P(168 < \bar{C} < 172) = 0.904$

Therefore, $n = 25$

Chapter Review

1 a $E(D) = 1.8 = 0 \times 0.3 + 1 \times (p+q) + 2 \times 0.15 + 3 \times (p-q) + 4 \times (p+2q) = 0.3 + 8p + 6q$ and $1 = 0.3 + p + q + 0.15 + p - q + p + 2q = 0.45 + 3p + 2q$

Simplifying $1.5 = 8p + 6q$ and $0.55 = 3p + 2q$

solving simultaneously gives $p = 0.15$ and $q = 0.05$

b $P(D = 3 | D > 1) = \dfrac{P(D=3)}{P(D>1)} = \dfrac{0.1}{0.5} = 0.2$

2 a

t	4	5	6	7	8	9	10
$P(T = t)$	$\frac{1}{36}$	$\frac{1}{9}$	$\frac{2}{9}$	$\frac{5}{18}$	$\frac{2}{9}$	$\frac{1}{9}$	$\frac{1}{36}$
Payment	x	-5	0	-7	0	x	0

b $\frac{1}{36} \times x + \frac{(-5)}{9} + \frac{(-7) \times 5}{18} + \frac{1}{9} \times x = 0 \Rightarrow x = 18$

3 a Factors of 120: $1 \times 120 = 120$, $2 \times 60 = 120$, $3 \times 40 = 120$, $4 \times 30 = 120$, $5 \times 24 = 120$, $6 \times 20 = 120$, $8 \times 15 = 120$, $10 \times 12 = 120$ so therefore 16 factors

b let F be the total number of numbers generated which are factors of 120. $F \sim B\left(6, \frac{16}{120}\right)$

i $P(F = 3) = 0.0309$

ii $P(F \geq 3) = 1 - P(F \leq 2) = 0.0346$

iii $P(F \leq 3) = 0.996$

iv There are 4 ways to get 3 consecutive factors of 120: If f is a number which is a factor of 120 the ways are $f\,f\,f\,f'\,f'\,f'$, $f'\,f\,f\,f\,f'\,f'$, $f'\,f'\,f\,f\,f\,f'$, $f'\,f'\,f'\,f\,f\,f$

$P(\text{ factor of 120 on 3 consecutive rolls}) = 4 \times \left(\frac{16}{120}\right)^3 \left(1 - \frac{16}{120}\right)^3 = 0.00617$

4 a $P(A \geq 3) = 1 - P(A \leq 2) = 0.123$

b $P((A_1 + A_2) = 3) = 0.210$

c $P(A = 0) = 0.2982$,

Number of days with no accidents in a week is $W \sim B(7, 0.2982)$

P(more than 4 days with no accidents)

$= P(W > 4) = 1 - P(W \le 4) = 0.0281$

5 **a** $P(W > 70) = 0.325$

b Let $Q_1 = a$ and $Q_3 = b$ $P(W < a) = 0.25 \Rightarrow a = 57.58$, using the inverse normal function on the GDC

$P(W < b) = 0.75 \Rightarrow b = 72.42$

$IQR = 72.42 - 57.58 = 14.8$

c Need to find k such that $P(W > k) = 0.073 \Rightarrow k = 81.0$ using the inverse normal function on the GDC. On some GDCs it is necessary to first write as $P(W < k) = 0.927$

d Let N be the number of boys who weigh at least 70kg. $N \sim B(8, 0.325)$

$P(N \le 3) = 0.758$

e $1000 \times P(W < 60) = 325$

6 **a** $F - 3R \sim N\left(430 - 3\times160, 9^2 + 9\times5^2\right)$

$P(F - 3R < 0) = 0.998$

b $F - R - R - R \sim N\left(430 - 3\times160, 9^2 + 3\times5^2\right)$

$P(F - R - R - R > 0) = 0.0000313$

7 Let Y be the value where the ball touches the y-axis. $Y = 2F + 3 \sim N\left(7.6, 4\times0.3\right)$

$E(P) = P(5 < Y \le 6)\times1 + P(6 < Y \le 8)\times5 + P(8 < Y \le 9)\times10$
$= 5.484...$

$127 \times E(P) = 127 \times 5.484... = 696$

8 **a** $E(C) = E(A - B) = E(A) - E(B) = 3.1 - 4.7 = -1.6$ and $\text{Var}(C) = \text{Var}(A - B) = \text{Var}(A) + \text{Var}(B) = 3.1 + 4.7 = 7.8$ so C is not Poisson because $E(C) \ne \text{Var}(C)$.

D is not Poisson as its domain includes only values greater than or equal to 9.

E is Poisson because $E(E) = E(A + B) = E(A) + E(B) = \text{Var}(A) + \text{Var}(B) = \text{Var}(A + B) = \text{Var}(E)$

$E(F) = E(2.9B) = 2.9E(B) = 13.63$ and $\text{Var}(F) = \text{Var}(2.9B) = 2.9^2\text{Var}(B) = 39.527$ so F is not Poisson because $E(F) \ne \text{Var}(F)$.

b Here, $B \sim \text{Po}(7)$. Need to find k such that $P(B < k) = 0.99$. If $k = 13$ then $P(B < k) = 0.987$, but if $k = 14$ then $P(B < k) = 0.994$ so 14 batteries should be in stock.

c Need to find λ such that
$P(U > 2) = 1 - P(U \le 2) = 0.401$,

This can be entered as an equation into the GDC and solved to give $\lambda = 2.2888$

9 a $E(T) = E(X + Y + 2Z) = E(X) + E(Y) + 2E(Z) = 3.07 + 5 \times 0.27 + 2 \times 3.81 = 12.04$

$\mathrm{Var}(T) = \mathrm{Var}(X + Y + 2Z)$

$= \mathrm{Var}(X) + \mathrm{Var}(Y) + 4\mathrm{Var}(Z) = 0.8^2 + 5 \times 0.27 \times 0.73 + 4 \times 3.81 = 16.9$

b By the central limit theorem, $\bar{T} \sim N\left(12.04, \dfrac{16.9}{35}\right)$

$P(11 \le \bar{T} < 13) = 0.850$

c This can be solved by educated guessing or by putting a function similar to following into the GDC and using the table to find the value of n.

$$F = normcdf\left(13, 12.04, \frac{16.9}{\sqrt{x}}\right) - normcdf\left(11, 12.04, \frac{16.9}{\sqrt{x}}\right)$$

If $n = 65$, $P(11 \le \bar{T} < 13) = 0.9497$ and if $n = 66$, $P(11 \le \bar{T} < 13) = 0.9514$

Exam style questions

10 i $E(X) = 0 = \dfrac{(-3)\times 1}{20} + \dfrac{(-2)\times 2}{20} + (-1)\times a + 0 \times b + \dfrac{1\times 3}{20} + \dfrac{2\times 2}{20} + \dfrac{3 \times 1}{20}$

$a = 0.15$

ii $1 = \dfrac{1}{20} + \dfrac{2}{20} + a + b + \dfrac{3}{20} + \dfrac{2}{20} + \dfrac{1}{20}$

$b = 0.4$

11 a $P(X = 5) = \dfrac{5^5 e^{-5}}{5!} = 0.1755$

b $P(X < 5) = \dfrac{5^0 e^{-5}}{0!} + \cdots + \dfrac{5^4 e^{-5}}{4!} = 0.4405$

c $W \sim \mathrm{Po}(7 \times 5)$, $P(W > 33) = 1 - P(W \le 33) = 1 - \left(\dfrac{35^0 e^{-35}}{0!} + \cdots + \dfrac{35^{33} e^{-35}}{33!}\right) = 0.5898$

d $P(X + Y \le 8) = \dfrac{9^0 e^{-9}}{0!} + \cdots + \dfrac{9^8 e^{-9}}{8!} = 0.4557$

e i $E(Z) = E(3X + 2Y) = 3E(X) + 2E(Y) = 3 \times 5 + 2 \times 4 = 23$

ii $\mathrm{Var}(Z) = \mathrm{Var}(3X + 2Y) = 3^2\mathrm{Var}(X) + 2^2\mathrm{Var}(Y) = 9 \times 5 + 4 \times 4 = 61$

f Does not satisfy Poisson distribution as $E(Z) \ne \mathrm{Var}(Z)$

12 a $S \sim B(10, 0.25)$

$P(S = 3) = \dbinom{10}{3}\left(\dfrac{1}{4}\right)^3\left(\dfrac{3}{4}\right)^7 = 0.2503$

b $S \sim B(100, 0.25)$

$P(S \le 27) = \dbinom{100}{0}\left(\dfrac{1}{4}\right)^0\left(\dfrac{3}{4}\right)^{100} + \cdots + \dbinom{100}{27}\left(\dfrac{1}{4}\right)^{27}\left(\dfrac{3}{4}\right)^{73} = 0.7224$

c $S \sim B(n, 0.25)$

$P(S \geq 50) = 1 - P(S \leq 49) = 0.75$

This can be etered as an equation into the GDC and the table function used to find the value of n

If $n = 215$, $P(S \geq 50) = 0.7460$ and if $n = 216$, $P(S \geq 50) = 0.7582$ so $n = 216$

13 a $T \sim N\left(3\times 60,\ 3^2 \times 3^2\right)$

$P(T > 175) = 0.711$

b

$$E(H - 2.5M) = 160 - 2.5 \times 60 = 10$$
$$\mathrm{Var}(H - 2.5M) = \mathrm{Var}(H) + 2.5^2\,\mathrm{Var}(M) = 5^2 + 2.5^2 \times 3^2 = 81.25$$

$P(H - 2.5M < 0) = 0.1336$

14 a $P(E > 55) = 0.1057 \Rightarrow 10.57\%$

b Let L be the number of large eggs in a box. $L \sim B(6,\ 0.1057)$
$P(L \geq 1) = 1 - P(L = 0) = 0.488$

15 Enter both expressions into the GDC and use the equation solving function

$P(X = 9) = \frac{1}{2}P(X = 7)$
$\mu = 6$

16 a **i** $E(X) = 1 \times \frac{1}{4} = \frac{1}{4}$ **ii** $\mathrm{Var}(X) = 1 \times \frac{1}{4} \times \frac{3}{4} = \frac{3}{16}$

b **i** $E(\bar{X}) = E(X) = \frac{1}{4}$ **ii** $\mathrm{Var}(\bar{X}) = \frac{\mathrm{Var}(X)}{N} = \frac{\frac{3}{16}}{100} = \frac{3}{1600}$

c $P(\bar{X} > 0.305) = 0.1020$

d $P\left(\sum_{1}^{100} X_i > 30.5\right) = P(\bar{X} > 0.305) = 0.1020$

e **i** $T \sim \mathrm{B}\left(100, \frac{1}{4}\right)$

ii $P(T > 30.5) = P(T \geq 31) = 1 - P(T \leq 30) = 0.1038$

17 a $P(X \leq 2) = P(X = 1) + P(X = 2) = \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$

b $E(X) = 1 \times \frac{1}{2} + 2 \times \frac{1}{3} + 3 \times \frac{1}{6} = \frac{5}{3}$

c **i** $E(Y) = E(4X) = 4E(X) = \frac{20}{3}$ **ii** $\mathrm{Var}(Y) = \mathrm{Var}(4X) = 4^2\mathrm{Var}(X) = \frac{80}{9}$

d **i** $E(T) = E(X_1 + X_2 + X_3) = E(X) + E(X) + E(X) = 3E(X) = 5$

ii $\text{Var}(T) = \text{Var}(X_1 + X_2 + X_3)$
$= \text{Var}(X) + \text{Var}(X) + \text{Var}(X)$
$= 3\text{Var}(X) = \frac{5}{3}$

e **i** $E(R) = \frac{1}{1} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{3} + \frac{1}{3} \times \frac{1}{6} = \frac{13}{18}$

ii Not true because $\frac{13}{18} \neq \frac{1}{\frac{5}{3}} = \frac{3}{5}$

14 Testing for validity: Spearman's, hypothesis testing and χ^2 test for independence

Skills check

1 $P(S) = \dfrac{n(\{1,4\})}{n(\{1,2,3,4,5,6\})} = \dfrac{1}{3}$, $P(E) = \dfrac{n(\{2,4,6\})}{n(\{1,2,3,4,5,6\})} = \dfrac{1}{2}$,

$P(E \cap S) = \dfrac{n(\{4\})}{n(\{1,2,3,4,5,6\})} = \dfrac{1}{6} = \dfrac{1}{3} \times \dfrac{1}{2} = P(S) \times P(E)$ so independent

2 $P(W < 36) = 0.6306$

3. Using the function on the GDC

$r = 0.7719$

Exercise 14A

1 a 1 because the data is strictly increasing

b 0.99 because the data is strictly increasing other than the first two points which are non-decreasing

c −1 because the data is strictly decreasing

d Approximately 0 because though there is a relation it is neither increasing or decreasing

2 a

x	y	R_x	R_y
0	23	1	7
5	18	2	6
10	10	3	5
15	9	4	4
20	7	5	2
25	7	6	2
30	7	7	2

Find the product moment correlation coefficient (r) of the ranls

$r_s = -0.9636$

b

x	y	R_x	R_y
10	12	5	6
12	11	6	5
9	8	4	3
6	5	2	1
3	7	1	2
14	14	7	7
8	9	3	4

$r_s = 0.8929$

3 a Not appropriate as the plot indicates that the relationship is not linear

b

x	y	R_x	R_y
0.25	30.4	1	10
0.51	25.1	2	9
0.69	20	3	8
1.09	15.6	4	7
1.52	13.4	5	5.5
2.02	13.4	6	5.5
2.43	11.2	7	4
2.67	10.2	8	1
2.93	10.8	9	3
3.17	10.3	10	2

$r_s = -0.9605$

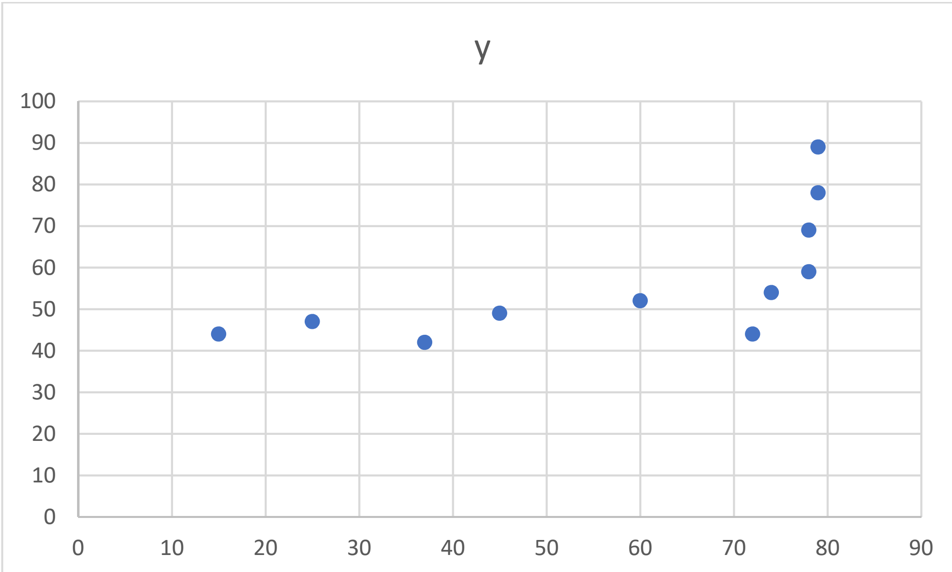
c The result indicates that there is a strong inverse relationship between velocity and force.

4 a

$r = 0.6699$

Moderate positive correlation. This indicates generally students who do better in Maths tend to do better in English.

b



Graph indicates that the data does not follow a linear model throughout the whole range of scores. Those with a high score in Maths have very variable scores in English. This is possibly due to the Mathematics test not separating students at the higher ability level, or students who are able at Mathematics not having English as a mother tongue. A linear model is probably not appropriate over the whole range.

c

x	y	R_x	R_y
15	44	1	2.5
25	47	2	4
37	42	3	1
45	49	4	5
60	52	5	6
72	44	6	2.5
74	54	7	7
78	59	8.5	8
78	69	8.5	9
79	78	10.5	10
79	89	10.5	11

Repeated ranks, so need to use repeated rank formula:

$r_s = 0.8833$

The Spearman's correlation coefficient is higher and indicates more strongly that those who do better in Maths do better in English. This is supported by the graph.

d Spearman's is the more useful measure of correlation because the relationship between the variables is not linear.

5 a Because the ranks are given rather than quantifiable data.

b

y	R_x	R_y
450	1	6
360	2	4
390	3	5
320	4	2
350	5	3
300	6	1

$r_s = -0.8857$

Generally, the preferred coffees are the more expensive ones.

6 a i $r = 0.8743$

$r = 0.0776$

b i

x	y	R_x	R_y
0.82	0.86	1	4
1.28	1.56	2	7
1.78	1.22	4	5
1.46	0.62	3	1.5
2.46	0.84	6	3
2.48	1.76	7	8
2.02	1.82	5	9
3.02	1.42	9	6
2.98	0.62	8	1.5
7.46	4.98	10	10

$r_s = 0.304$

ii $r_s = 0.0418$

c Spearman's correlation coefficient is affected less by an outlier than the PMCC.

Exercise 14B

1 a Assume that H_0 is true, then

$P(X \geq 7) = \binom{12}{7} \times 0.3^7 \times 0.7^5 + \quad + \binom{12}{12} \times 0.3^{12} \times 0.7^0 = 0.03860 < 0.05$ this is significant, so

therefore reject H_0

b Assume that H_0 is true, then

$P(X \le 6) = \binom{20}{0} \times 0.4^0 \times 0.6^{20} + \quad + \binom{20}{6} \times 0.4^6 \times 0.6^{14} = 0.2500 \nless 0.1$ this is not significant, so therefore no reason to reject H_0

c Assume that H_0 is true, then $P(X \ge 9) = \binom{10}{9} \times 0.7^9 \times 0.3^1 + \binom{10}{10} \times 0.7^{10} \times 0.3^0 = 0.1493 \nless 0.1$ this is not significant, so therefore no reason to reject H_0

2 **a** $H_0 : p = 0.6$, $H_1 : p < 0.6$

b Need to find r such that $P(X \le r) = 0.05$. $r = 13 \Rightarrow P(X \le r) = 0.04811$, $r = 14 \Rightarrow P(X \le r) = 0.09706$ so the critical region is $X \le 13$

c As $14 \nleq 13$, the result is not significant, so therefore no reason to reject H_0. Thus, the treatment leads to a significant reduction in symptoms.

3 **a** Need to find r such that $P(X \ge r) = 0.05$. $r = 3 \Rightarrow P(X \ge r) = 0.05297$, $r = 4 \Rightarrow P(X \ge r) = 0.008331$ so the critical region is $X \ge 4$

b Binomial may not be appropriate as tripping on one fence may mean tripping on others (i.e. not independent)

4 $H_0 : p = 0.3$, $H_1 : p > 0.3$. Assume that H_0 is true, then

$P(X \ge 8) = \binom{20}{8} \times 0.3^8 \times 0.7^{12} + \quad + \binom{20}{20} \times 0.3^{20} \times 0.7^0 = 0.2277 \nless 0.05$ this is not significant, so therefore no reason to reject H_0. There is insufficient evidence to suggest that the course helped.

5 $H_0 : p = 0.1$, $H_1 : p > 0.1$. Assume that H_0 is true, then

$P(X \ge 2) = \binom{5}{2} \times 0.1^2 \times 0.9^3 + \quad + \binom{5}{5} \times 0.1^5 \times 0.9^0 = 0.08146 \nless 0.05$ this is not significant, so therefore no reason to reject H_0. There is insufficient evidence to suggest that the bus is late more than 10% of the time.

Exercise 14C

1 **a** Assume that H_0 is true, then $P(X \le 5) = 0.2759 > 0.05$ this is not significant, so therefore no reason to reject H_0

b Assume that H_0 is true, then $P(X \ge 10) = 0.07721 > 0.05$ this is not significant, so therefore no reason to reject H_0

c Assume that H_0 is true, then $P(X \ge 23) = 1 - P(X \le 22) = 0.005876 < 0.01$, this is significant, so therefore reject H_0

2 $H_0 : \lambda = 2 \times 1.8 = 3.6$, $H_1 : \lambda < 3.6$. Assume that H_0 is true, then $P(X \le 2) = 0.3028 > 0.05$. This is not significant, so therefore no reason to reject H_0. There is insufficient evidence to suggest that the bus is late more than 1.8 times per week.

3 **a** $H_0 : \lambda = 5 \times 2.2 = 11$, $H_1 : \lambda < 11$

b Need to find r such that $P(X \le r) = 0.05$. $r = 5 \Rightarrow P(X \le r) = 0.03752$, $r = 6 \Rightarrow P(X \le r) = 0.07861$ so the critical region is $X \le 5$. The actual significance level for the test is 3.75%

c A constant rate is assumed for the Poisson distribution, which may not be appropriate as the number of goals conceded depends on the strength of the opposition, the players available, the weather, etc.

4 **a** $H_0 : \lambda = 0.25 \times 16 = 4.0$, $H_1 : \lambda > 4.0$. Need to find r such that $P(X \ge r) = 0.05$
$r = 8 \Rightarrow P(X \ge r) = 0.05113$, $r = 9 \Rightarrow P(X \ge r) = 0.02136$ so the critical region is $X \ge 9$.

b Need to assume that the infections occur independently of each other.

Exercise 14D

1 **a** **i** $r = 0.7206$

ii p-value $= 0.4878 > 0.1$ so no reason to reject the null hypothesis so there is insufficient evidence that there is a linear correlation.

b **i** $r = 0.7246$

ii p-value $= 0.1033 > 0.1$ so no reason to reject the null hypothesis so there is insufficient evidence that there is a linear correlation

c **i** $r = 0.7176$

ii p-value $= 0.008597 < 0.1$ so reason to reject the null hypothesis that there is no correlation between the two variables

iii $y = 42.7981 + 0.5205x$

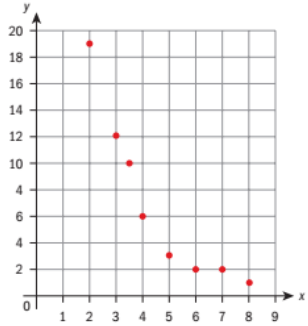
2 **a** $r = -0.9013$

b **i** p-value $= 0.00112 < 0.05$, which is highly significant and so the null hypothesis is rejected and the alternative hypothesis that there is a negative correlation between price and number of sales is accepted.

ii $s = 20.2 - 2.76p$

iii $s = 20.2 - 2.76p \Rightarrow s = 20.2 - 2.76 \times 5.50 = \$5.02$

c



The relationship is not linear, so even though the p-value was very small, the linear model may not be appropriate.

3 **a** $r = 0.8683$

p-value $= 0.001119 < 0.05$, which is highly significant and so the null hypothesis is rejected and the alternative hypothesis that there is a positive correlation between sales and temperature is accepted.

b Let x be the temperature (the independent variable) and y be the ice cream (the dependent variable). $y = 3.94x + 78.1$

c $y \approx 3.94 \times 23 + 78.1 \Rightarrow y \approx €169$

d 35°C is outside the domain of the data provided, and it is not wise to extrapolate.

Exercise 14E

1 These questions should be answered using the Z-test function on your GDC

a p-value = 0.166 > 0.05 not significant, so do not reject H_0

b p-value = 0.0302 < 0.05 so significant, reject H_0

c p-value = 0.412 > 0.10 not significant, so do not reject H_0

2 **a** The sample size is large enough for the central limit theorem to apply

b $H_0 : \mu = 24$, $H_1 : \mu < 24$

c p-value $= 0.01391 < 0.05$, significant, so enough evidence to reject H_0 that the amount of area covered is 24 m^2

d Need to find r such that $P(\bar{X} \le r) = 0.05$.

Under H_0 $\bar{X} \sim N\left(24, \dfrac{1.8^2}{32}\right)$

To find the critical value use the inverse normal function on your GDC with the appropriate mean and standard deviation

$P(\bar{X} \le r) = 0.05 \Rightarrow r = invnorm(0.05) \Rightarrow r = 23.48$, so the critical region is $\bar{X} \le 23.48$, thus confirming the result of the test as we had a value of $23.3 \le 23.48$

3 **a** $H_0 : \mu = 8.3$, $H_1 : \mu > 8.3$

b $\bar{x} = \dfrac{8 + 8.7 + 9.2 + 8.4 + 8.5}{5} = 8.56$

c $\bar{X} \sim N\left(8.3, \dfrac{2.1^2}{5}\right) = N(8.3, 0.882)$

d $P(\bar{X} > \bar{x}) = P(\bar{X} > 8.56) = 0.391$

e $0.391 > 0.1$ so not enough evidence to reject the null hypothesis at the 10% significance level that the mean time between buses is 8.3 minutes.

f p-value = 0.391

g Need to find r such that $P(\bar{X} \geq r) = 0.05$, using the inverse normal function on the GDC $P(\bar{X} \geq r) = 0.05 \Rightarrow P(\bar{X} < r) = 0.95 \Rightarrow r = invnorm(0.95) \Rightarrow r = 9.8448$, so the critical region is $\bar{X} \geq 9.8448$

Exercise 14F

These questions should be done using the t-test function on the GDC

1 **a** p-value $= 0.1001 > 0.05$, so not significant so do not reject H_0

b p-value = 0.421 > 0.05 not significant, so do not reject H_0

c p-value = 0.0239 < 0.05 significant, so reject H_0

2 **a** p-value = 0.058 > 0.05 not significant, so do not reject H_0

b p-value = 0.066 > 0.05 not significant, so do not reject H_0

c p-value = 0.0952 < 0.10 significant, reject H_0

3 **a** It is possible because the sample size is large enough for the central limit theorem to apply

b $H_0 : \mu = 28.2$, $H_1 : \mu > 28.2$

c p-value = 0.233 > 0.05 not significant, so insufficient evidence

4 **a** $H_0 : \mu = 83$, $H_1 : \mu > 83$

b i 86.0 ii 3.41 iii 3.81

c 0.0764 > 0.05, not significant so insufficient evidence to reject the null hypothesis that the mean journey time is 83 minutes.

Exercise 14G

These questions should be done using the Z or T confidence interval function on the GDC

1 **a** (10.4,14.4)

b (61.9,62.7)

c Note that if your GDC requires you to enter the unbiased estimator you need to enter $\sqrt{\dfrac{10 \times 5.2}{9}}$ (4.58,8.02)

d Note that if your GDC requires you to enter the unbiased estimator you need to enter $\sqrt{\dfrac{10 \times 0.2^2}{9}}$ (2.10,2.50)

e As the population standard deviation is known find the Z (normal) interval rather than the T interval (4.47,4.73)

2 These are small samples so use Student's *t*.

a (10.24,14.28)

b (−7.85,6.89)

3 **a** **i** (19.76,22.64)

ii (20.30,22.10)

iii (20.67,21.73)

iv (20.83,21.57)

b The larger the sample the smaller the width of the confidence interval

4 **a** (11.57,15.23)

b That the population can be modelled by a normal distribution

c 15.3 is outside the range for the confidence interval so is unlikely to be the population mean.

Exercise 14H

In this exercise use the two sample T-test function on your GDC

1 $H_0 : \mu_1 = \mu_2$, $H_1 : \mu_1 < \mu_2$

p-value = 0.251

0.251 > 0.10, not significant, do not reject the null hypothesis. There is no difference in the weights of the apples.

2 $H_0 : \mu_1 = \mu_2$, $H_1 : \mu_1 > \mu_2$

p-value = 0.00539

0.00539 < 0.01, significant, so reject the null hypothesis. Those using the new remedy do lose more weight.

Exercise 14I

1

Differences	7	1	7	7	8	9	4	−5	5	6

$H_0 : \mu_D = 0$, $H_1 : \mu_D > 0$,

p-value 0.00233 < 0.01, significant so reject H_0 that there has been no improvement.

2 **a** $H_0 : \mu_D = 5$, $H_1 : \mu_D > 5$

b

Differences	5	10	6	8	3	−1

p-value 0.460 > 0.05, not significant so insufficient evidence to reject H_0 that the company on average increases a client's investment by 5%.

3 **a**

Differences	−0.6	−1.7	−0.2	−1.5	−1.7	−2	−1.4	−0.7	−0.6	−0.8

$H_0 : \mu_D = 0$, $H_1 : \mu_D < 0$

p-value = 0.000127<0.05 so the result is significant and the drug has a positive effect.

b $H_0 : \mu_D = 0.7$, $H_1 : \mu_D < 0.7$

p-value $= 0.0285 < 0.05$, so the result is significant and the drug has a positive effect on top of the healthy diet.

c **i** Yes because there is some good evident the drug is working.

ii Larger sample, group chosen to eliminate other factors e.g. age, gender.

Exercise 14J

1 $H_0 :$ favourite flavour of chocolate is independent of gender

$H_1 :$ favourite flavour of chocolate is not independent of gender

From the inbuilt function on the GDC

Either: the χ^2 test statistic = 9.52. 9.52 > 9.210, significant so reject the null hypothesis that favourite chocolate and gender are independent.

Or: the p-value = 0.00856. 0.00856 < 0.01, significant so reject the null hypothesis that favourite chocolate and gender are independent.

2 **a** $H_0 :$ GPA is independent of number of hours on social media

$H_1 :$ GPA is not independent of number of hours on social media

b $\frac{85}{270} \times \frac{99}{270} \times 270 = 31.1667 \approx 31.2$

c Degrees of freedom $= (3-1) \times (3-1) = 2 \times 2 = 4$

d The χ^2 test statistic = 78.5 and the p-value = 3.59×10^{-16}.

The critical value is 7.779.

e Either: 78.5 > 7.779,

Or: 3.59×10^{-16} < 0.10

Significant so reject the null hypothesis that GPA is independent of number of hours on social media.

3 $H_0 :$ number of people walking their dog is independent of the time of day

$H_1 :$ number of people walking their dog is not independent of the time of day

Either: The χ^2 test statistic = 5.30

5.30 < 9.488 so the result is not significant so no reason to reject the null hypothesis that number of people walking their dog is independent of the time of day.

Or: the p-value = 0.257

0.257 > 0.05 so the result is not significant so no reason to reject the null hypothesis that number of people walking their dog is independent of the time of day.

4 **a** $H_0 :$ type of degree that a person has is independent of their annual salary

$H_1 :$ type of degree that a person has is not independent of their annual salary

b For the cell containing the number of people with a BA who earn less than \$60,000:

$\frac{18}{100} \times \frac{27}{100} \times 100 = 4.86 < 5$

c

Observed	BA	MA	PhD
< \$120 000	20	23	13
> \$120 000	7	13	24

d p-value $= 0.00403$

e $0.00403 < 0.01$ The result is significant so reject the null hypothesis that type of degree and salary are independent.

Exercise 14K

1 a

	Yellow	Orange	Red	Purple	Green
Observed	104	132	98	129	137
Probability	0.2	0.2	0.2	0.2	0.2
Expected	120	120	120	120	120

b 4 degrees of freedom

c H_0 : the colours follow a uniform distribution

H_1 : the colours do not follow a uniform distribution

$\chi^2 =$ 10.45 and the p-value = 0.0335

Either 10.45 > 9.488 or 0.0335 < 0.05

The result is significant so reject the null hypothesis that the distribution is uniform.

2 a

	0	1	2	3	4	5	6	7	8	9
Observed	44	53	49	61	47	52	39	58	42	45
Probability	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1
Expected	50	50	50	50	50	50	50	50	50	50

b H_0 : the last number follows a uniform distribution

H_1 : the last number does not follow a uniform distribution

The p-value = 0.430

0.430 > 0.10 not significant so no reason to reject the null hypothesis that the last number on the lottery tickets follows a uniform distribution.

3 a

	$x < 50$	$50 \le x < 60$	$60 \le x < 70$	$70 \le x < 80$	$80 \le x$
Probability	0.02275	0.2297	0.4950	0.2297	0.02275
Expected	6.83	68.92	148.50	68.92	6.83

b H_0 : the grades fit a normal distribution with mean 65% and s.d. of 7.5%.

H_1 : the grades don't fit a normal distribution with mean 65% and s.d. of 7.5%.

The p-value = 0.947

0.947>0.10 so the result is not significant and hence there is no reason to reject the null hypothesis that the exam results follow the given distribution.

4 a

	$h < 235$	$235 \le h < 245$	$245 \le h < 255$	$255 \le h < 265$	$265 \le h$
Probability	0.08634	0.2384	0.3506	0.2384	0.08634
Expected	21.6	59.6	87.6	59.6	21.6

b H_0 : the heights fit a normal distribution with mean 250 and s.d. of 11.

H_1 : the heights don't fit a normal distribution with mean 250 and s.d. of 11.

p-value = 0.0906

0.0906 > 0.05 not significant so no reason to reject the null hypothesis that the sample of elephants was taking from a population whose heights are fit a $N\left(250, 11^2\right)$ distribution.

5 a H_0 : the scores are normally distributed with mean of 100 and s.d. of 10.

H_1 : the scores are not normally distributed with mean of 100 and s.d. of 10.

b Because if the expected values for the scores outside the range of the observed data are not zero then they would contribute to the test statistic.

c

	$x < 90$	$90 \le x < 100$	$100 \le x < 110$	$110 \le x < 120$	$120 \le x < 130$	$130 \le x$
Probability	0.1587	0.3413	0.3413	0.1359	0.02140	0.00135
Expected	31.73	68.27	68.27	27.18	4.28	0.27

d

	$x < 90$	$90 \le x < 100$	$100 \le x < 110$	$110 \le x$
Observed	18	39	78	65
Probability	0.1587	0.3413	0.3413	0.1587
Expected	31.73	68.27	68.27	31.73

e 3 degrees of freedom

f $\chi^2_{calc} = 54.8$ and p-value= 7.40×10^{-12}

58.4 > 11.3 or $7.40 \times 10^{-12} < 0.01$ hence the result is significant and the null hypothesis that the IQs of the students have been taken from this distribution should be rejected.

g **i** Mean $= 105$
Standard deviation=
$= 13.56$

ii The data is fairly symmetrical and most of the data is within two standard deviations of the mean which indicates a normal distribution is possible. The high value of the chi-squared statistic is probably due to the mean and the standard deviation being somewhat higher for those being tested.

iii Redo the test with a different mean and standard deviation.

Exercise 14L

1 **a** **i** This is a binomial distribution with *n*=2 and $p = \frac{1}{6}$

Number of sixes	**0**	**1**	**2**
Probability	0.694	0.278	0.0278
Expected	173.61	69.44	6.94

ii In table above

b H_0 : the dice are fair

H_1 : the dice aren't fair

$X^2 = 28.1461 > 5.991$. The result is significant so there is strong evidence to reject the null hypothesis that the dice are fair.

2 **a**

There will be several methods for generating these values on your GDC. For example entering the Poisson probability density function (*Poissonpdf* or similar) as a function and using the table function to read off the values. The expected frequency is the probability multiplied by 50

Goals	**Probability**	**Expected Frequency**
0	0.0907	4.53
1	0.218	10.89
2	0.261	13.07
3	0.209	10.45
4	0.125	6.27
5	0.0602	3.01
≥6	0.0357	1.78

b H_0 : the number of goals follows a Poisson distribution with a mean of 2.4

H_1 : the number of goals doesn't follow a Poisson distribution with a mean of 2.4

Need to first combine the final three rows of the table.

≥4	0.22128	11.06

Goals	0	1	2	3	≥4
Observed	7	10	11	10	12
Expected	4.53	10.89	13.07	10.45	11.06

$X^2 = 1.8299$, p-value=0.767>0.1. The result is not significant and so there is insufficient evidence to reject the null hypothesis that the sample is taken from a population that follows a Po(2.4) distribution.

3 a Using the probability density function on the GDC for B(3, 0.75)

Seeds	**Probability**	**Expected Frequency**
0	0.0156	0.78
1	0.1406	7.03
2	0.4219	21.09
3	0.4219	21.09

b H_0 : the number of seeds germinating fit a B(3.0.75) distribution

H_1 : the number of seeds germinating doesn't fit a B(3.0.75) distribution

c Need to combine first two columns of the table.

Seeds germinating	0 or 1	2	3
Observed	15	15	20
Probability	0.1562	0.4219	0.4219
Expected	7.81	21.09	21.09

d 2 degrees of freedom

e p-value $= 0.01474 < 0.05$.

The result is significant so reject the null hypothesis that the seeds fit a B(3.0.75) distribution.

4 a No of 5-minute periods = $\frac{5 \times 60}{5} = 60$

Generate the Poisson probabilities using the inbuilt functions on the GDC then multiply by 60 to obtain the expected values.

Number of People	Probability	Expected Frequency
0	0.0150	0.90
1	0.0630	3.78
2	0.1323	7.94
3	0.1852	11.11
4	0.1944	11.66
5	0.1633	9.80
6	0.1143	6.86
7	0.0686	4.12
≥8	0.0639	3.84

Combine the first three and last two rows:

≤2	$0.0150+0.0630+0.1323=0.2103$	12.62
≥7	$0.0686+0.0639=0.1325$	7.96

p-value $=0.00308<0.05$

The result is significant so reject the null hypothesis that the number of people joining the queue follows a Po(4.2) distribution.

b It is assumed that the average number of people arriving for the cell ≥8 people is 8.

i Mean $=4.23$

Obtain the value for the standard deviation from the GDC then square it to obtain the variance $=6.81$

ii The parameter for the Poisson is the mean and the mean of the sample is close to the mean for the distribution so it is likely a correct mean was chosen.

For the Poisson distribution the mean and variance of the sample should be similar, that is not the case here which indicates the distribution is not Poisson.

5 **a** $X\sim \mathrm{B}(5,0.25)$

b

X	Probability	Expected Frequency
0	0.2373	118.65
1	0.3955	197.75
2	0.2637	131.84
3	0.0879	43.95

4	0.0146	7.32
5	0.0010	0.49

c Need to combine the final two rows of the table

4 or 5	$0.0146+0.0010=0.0156$	7.81

H_0 : the students are guessing randomly

H_1 : the students are not guessing randomly

There are 4 degrees of freedom. The p-value = 0.0495

0.0495 < 0.05 just significant so some evidence to reject the null hypothesis that the students are guessing randomly.

Exercise 14M

1 a

Standard deviation
$=121.22\approx 121$

b Using the N(1200, 121^2) distribution

Lifespan, *h* hours	**Probability**	**Expected Frequency**
$h<1000$	$P(H<1000)=0.0492$	19.7
$1000\le h<1100$	$P(1000\le H<1100)=0.1551$	62.0
$1100\le h<1200$	$P(1100\le H<1200)=0.2957$	118.3
$1200\le h<1300$	$P(1200\le H<1300)=0.2957$	118.3
$1300\le h<1400$	$P(1300\le H<1400)=0.1551$	62.0
$1400\le h$	$P(H\ge 1400)=0.0492$	19.7

c Degrees of freedom: $6-1-1=4$

d H_0 : the lifespan of lightbulbs follows a $N(1200,121^2)$ distribution

H_1 : the lifespan of lightbulbs doesn't follow a $N(1200,121^2)$ distribution

$p\text{-value}=5.83\times 10^{-7}<0.05$. This result is significant and so we reject the null hypothesis that the data follows a $N(1200,121^2)$ distribution.

2 a $\text{Mean}=np=1.5319$, so as $n=3\Rightarrow p=0.5106$

b H_0 : the number of boys in a family has a B(3,0.5106) distribution

H_1 : the number of boys in a family doesn't have a B(3,0.5106) distribution

Number of boys	**Probability**	**Expected Frequency**
0	0.1172	11.72
1	0.3669	36.69

2	0.3828	38.28
3	0.1331	13.31

No of degrees of freedom = 4 − 1 − 1 = 2.

p-value = 0.0732 > 0.01. The result is not significant and so there isn't enough evidence to reject the null hypothesis that the number of boys in these families follows a B(3,0.5106) distribution.

3 **a** Mean $= \dfrac{0\times 7+1\times 10+2\times 15+3\times 21+4\times 14+5\times 9+6\times 4}{7+10+15+21+14+9+4} = 2.85$

b H_0 : the number of fish caught has a Po(2.85) distribution

H_1 : the number of fish caught doesn't have a Po(2.85) distribution

Number of fish	Probability	Expected Frequency
0	0.0578	4.6
1	0.1649	13.2
2	0.2349	18.8
3	0.2232	17.9
4	0.159	12.7
5	0.0906	7.3
≥6	0.0696	5.6

Need to combine the first two rows:

0 or 1	$0.0578 + 0.1649 = 0.2227$	17.8

Degrees of freedom = 6 − 1 − 1 = 4.

p-value $= 0.676 > 0.05$. The result is not significant so no reason to reject the null hypothesis that the number of fish caught follows a Po(2.85) distribution.

4 **a** Mean $= 52.6$

$s_{n-1} = 5.15$

b H_0 : the weights of the children has a $N(52.6, 5.15^2)$ distribution

H_1 : the weights of the children doesn't have a $N(52.6, 5.15^2)$ distribution

Weight, w kg	Probability	Expected Frequency
$w < 45$	$P(W < 45) = 0.07001$	14.00
$45 \le w < 50$	$P(45 \le W < 50) = 0.2368$	47.36
$50 \le w < 55$	$P(50 \le W < 55) = 0.3726$	74.51

$55 \le w < 60$	$P(55 \le W < 60) = 0.2452$	49.05
$60 \le w$	$P(W \ge 60) = 0.0754$	15.07

Degrees of freedom = 5 − 1 − 2 = 2

p-value $= 5.24 \times 10^{-5} < 0.05$. The result is significant so the sample is very unlikely to have come from a population with a $N(52.6, 5.15^2)$ distribution.

c The data has two peaks, which suggests two populations. For example, the sample may have included a mixture of boys and girls.

5 **a** $X \sim B(50, p)$

b $\text{Mean} = np = 1.9 \Rightarrow p = \frac{1.9}{50} = 0.038$

c H_0 : the sample is from a $B(50, 0.038)$ population

H_1 : the sample isn't from a $B(50, 0.038)$ population

Number of Prizes	Expected Frequency
0	8.65
1	17.08
2	16.53
3	10.45
4	4.85
≥5	2.45

Need to combine final two rows:

≥4	7.3

Degrees of freedom = 5 − 1 − 1 = 3.

p-value = 0.67 > 0.05

The result is not significant so no reason to reject the null hypothesis that the prizes are distributed randomly and independently.

6 **a** H_0 : the sample is from a Po(2.0) distribution

H_1 : the sample isn't from a Po(2.0) distribution

Number of 5-minute periods $= \frac{5 \times 60}{5} = 60$

Number of calls	Probability	Expected Frequency
0	0.1353	8.12

1	0.2707	16.24
2	0.2707	16.24
3	0.1804	10.82
4	0.0902	5.41
≥5	0.0527	3.16

Need to combine the final two rows:

≥4	$0.0902 + 0.0527 = 0.1429$	8.57

Degrees of freedom = 5 − 1 = 4.

p-value $= 0.0133 < 0.05$. This is significant, so reject the null hypothesis that the data follows a Po(2.0) distribution.

b Mean $= 2.57$

H_0 : the sample is from a Po(2.57) distribution

H_1 : the sample isn't from a Po(2.57) distribution

Number of calls	**Probability**	**Expected Frequency**
0	0.0765	4.59
1	0.1967	11.80
2	0.2528	15.17
3	0.2165	12.99
4	0.1391	8.35
≥5	0.1184	7.10

Degrees of freedom = 6 − 1 − 1 = 4.

p-value $= 0.335 > 0.05$. hence the result is not significant and there is no reason to reject the null hypothesis that the calls arriving at the business follow Po(2.57).

Exercise 14N

1 a The first data shows random error and the second shows systematic error. The second dataset follows $y = 1 + 2.2x$

b The second data would have a correlation of 1, the first data would be close to but not equal to 1.

c A high correlation normally means a line of regression is useful. In this case the systematic error has resulted in a perfect correlation but the line of regression would be increasingly inaccurate as x increases.

2 a Perform the χ^2 test for independence using the inbuilt function on your GDC

Expected	**0-10**	**11-20**	**21-30**

Positive	30.8	16.1	23.1
Negative	13.2	6.9	9.9

Degrees of freedom = (3 − 1) × (2 − 1) = 2

p-value $= 0.0300 < 0.05$ so significant and the two factors are not independent.

b Each of the 10 attributes can be paired with 9 others, to give 90 pairings but each pair will be counted twice so you need to divide by 2. Number of pairings is $\frac{10\times 9}{2}=45$ or you could reason the first attribute can be paired with 9 others, the second with 8 more and so on to give $9+8+7+\ldots+2+1=45$

c If all attributes are independent, then the probability of obtaining a significant result by chance is 0.05.
Let X be the number of pairs that yield significant results then $X \sim \mathrm{B}(45, 0.05)$ and $P(X \geq 1) = 1 - P(X = 0) = 0.9006$.

d Need to know if there are other reasons to suspect the two attributes given in part **a** were not independent. If there are no other reasons, need to do further tests as not enough evidence otherwise.

3 **a** **i** Taking a random sample or a sample stratified by gender. If measuring which school is better it is important the sample is representative of the school.

ii The ratio of boys to girls in each sample should be as equal as possible, so the improvement is due to the teaching method and not to the gender of the student.

iii A sample of girls should be compared with a sample of boys for each of the schools. The samples could also be pooled to give boys and girls from both schools. However, to avoid the results being affected by the teaching method, in the pooled sample, the ratio of boys to girls from each school should be equal.

b A t-test on the difference between the average improvement in each school to see if there is a significant difference. Assume the populations are normally distributed or the sample size is large enough for the central limit theorem to apply.

4 **a** Good: Any reason such as: The census contains details of most of the population. It will be relatively cheap and easy to collect. Because it contains other information, focused sampling could take place if required.

Bad: Any reason such as: The data is 6 years out of date. It records who was living in the house on census day so if taken during a school holiday it may include students who do not normally live at home.

b **i** Using $X \sim B(1000, 0.15)$,
$P(140 \leq X \leq 160) = P(X \leq 160) - P(X \leq 139) = 0.8242 - 0.1765 = 0.648$

ii $P(100 \leq X \leq 200) = P(X \leq 200) - P(X \leq 100) = 1.0000 - 2.039\times 10^{-6} = 1$
The survey is almost certain to find a proportion correct to within 50 households or 5%, but there is a about a 1 − 0.648 = 0.352 chance it will not be within 10 households or 1%. If this level of accuracy is needed, then more households will need to be surveyed.

c Do you have any children who have left school? If so, how many live at home and how many live away from home? (A single question such as ‘Do you have any children who are still living at home?' is ambiguous as it would be answered 'no' both by those with no children and by those with children who are not living at home.)

5 **a** The method of obtaining the sample, to ensure it is appropriate (the two groups should be as uniform as possible) and unbiased. Any confounding factors such as age, employment status or household income. Any assumptions made about the distributions, for example could the distribution be assumed to be normal.

b Let X be the number of significant results in the 8 tests. Then $X \sim \mathrm{B}(8, 0.05)$ and $P(X \geq 1) = 1 - P(X = 0) = 0.3366$

This means that finding one significant result is not in itself significant, so the result is not meaningful

Exercise 14O

1 **a** **i** Two outcomes, each trial independent and identical.

ii Events are independent and occur at a uniform average rate during the period of interest.

iii Data comes from a normal population or the sample size is large so that the central limit theorem applies

b Chi-squared goodness of fit test

2 **a** The Poisson distribution assumes that the number of infections occur independently and uniformly.

However, new infections are unlikely to be independent as the people already infected may come from the same wards or be receiving treatment from the same people.

The new infections are unlikely to be uniform, because over time more people will have been infected and the number of people susceptible will reduce so that the rate of new infections will decrease. The hospital may also take precautions such as isolating infected wards which would reduce the rate of infections.

b A chi-squared goodness of fit. A significant result might be because the distribution is not Poisson or because the rate of infection is not r .

3 **a** Chi-squared test or test for $\rho = 0$

b Chi-squared as we do not know if distance or weekly allowance are normally distributed. If it is suspected they might be normally distributed a test should be done to check this.

c The distances and allowances need to be categorised into groups so that the expected values are all greater than 5.

4 **a** Difficult to find parallel forms for questions about food. Test-retest will also indicate any change over time as well as considering the reliability of the data.

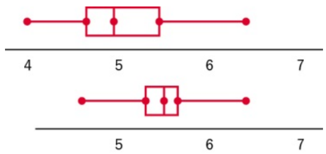
b $r = 0.907$
which indicates the survey is reliable as a source of information

c Using the statistical summary data from your GDC

Test 1 quartiles 4.5 and 5.5, median 4.9

Test 2 quartiles 5.2 and 5.7, median 5.5

Test 1:



d The box plots indicate there has been an improvement as the median and lower quartile have increased considerably, and the upper quartile has increased slightly though the maximum has not changed.

e Box plots are reasonably symmetrical so no reason to assume not normal, so reasonable to use the t-test.

Differences	0.8	0	0.4	0.6	0.8	0.6	0.6	−0.2

$H_0 : \mu_D = 0$, $H_1 : \mu_D > 0$

p-value $= 0.00518 < 0.05$, so the result is significant and there is significant evidence to say that the canteen has improved.

5 **a** $H_0 : \rho = 0$, $H_1 : \rho > 0$

p-value$= 0.00246$,

If testing $H_1 : \rho \neq 0$

p-value is $0.004916 < 0.05$.

In either case this is significant so very strong evidence to reject H_0 that there is no correlation between height and salary.

b The mix of males and females implies that heights are not likely to have been normal (i.e. they may be bimodal). Females on average are shorter than males, and this factor has not been considered. The positive relationship between height and salary might reflect that women are paid less than men on average. It would be better to stratify by gender and alter the hypothesis accordingly.

c Group the sample into salary bands and use a chi-squared test for independence between gender and salary or use a 2-sample t-test on average salary with the null hypothesis that there is no difference between the salaries of men and women.

Exercise 14P

1 **a** Under H_0 that $\mu = 7$, $P(X \geq a) = 0.05 \Rightarrow a = 7.12$

$P(X \leq 7.12 \mid \mu = 7.1) = 0.608$

b Under H_0 that $\mu = 12.1$, $P(X \leq a) = 0.05 \Rightarrow a = 12.026$,

$P(X \geq 12.026 \mid \mu = 12.0) = 0.28$

2 **a** **i** Need to find smallest a such that $P(X \geq a) \leq 0.05$. $P(X \geq 22) = 0.0940 > 0.05$,

$P(X \geq 23) = 0.0435 \leq 0.05$ so critical region is $X \geq 23$ and $P(X \geq 23 \mid p = 0.6) = 0.0435$

ii $P(X < 23 \mid p = 0.68) = 0.792$

b **i** Need to find largest a such that $P(X \leq a) \leq 0.05$. $P(X \leq 12) = 0.0386 \leq 0.05$

$P(X \leq 13) = 0.0751 > 0.05$ so critical region is $X \leq 12$ and $P(X \leq 12 \mid p = 0.45) = 0.0386$

ii $P(X > 12 \mid p = 0.43) = 0.935$

3 a i Here, $X \sim \text{Po}(30 \times 4.6) = \text{Po}(138)$.

Need to find largest a such that $P(X \le a) \le 0.05$. $P(X \le 118) = 0.0458 \le 0.05$
$P(X \le 119) = 0.0551 > 0.05$ so critical region is $X \le 118$ and $P(X \le 118 \mid \mu = 4.6) = 0.0458$

ii Here, $X \sim \text{Po}(30 \times 4.5) = \text{Po}(135)$.

$P(X > 118 \mid \mu = 4.5) = 0.925$

b i Here, $X \sim \text{Po}(20 \times 2.1) = \text{Po}(42)$.

Need to find smallest a such that $P(X \ge a) \le 0.05$. $P(X \ge 53) = 0.0566 > 0.05$
$P(X \ge 54) = 0.0422 \le 0.05$ so critical region is $X \ge 54$ and $P(X \ge 54 \mid \mu = 2.1) = 0.0422$

ii Here, $X \sim \text{Po}(20 \times 2.8) = \text{Po}(56)$.

$P(X < 54 \mid \mu = 2.8) = 0.377$

4 a Need to find a such that $P(b \le \bar{X} \le c) = 0.95$.

$P(\bar{X} \le b) = 0.025 \Rightarrow c = 48.3$
$P(\bar{X} \le c) = 0.975 \Rightarrow c = 53.7$. So critical regions are $\bar{X} < 48.3$ and $\bar{X} > 53.7$.

b $P(48.3 \le \bar{X} \le 53.7 \mid \mu = 51.5) = 0.9364$

5 a Under H_0 that $\mu = 50$, Need to find a such that $P(\bar{X} \ge a) = 0.05$.
$P(\bar{X} \ge a) = 0.05 \Rightarrow a = 53.121$. So critical region is $\bar{X} > 53.121$

b $P(\bar{X} < 53.121 \mid \mu = 54) = 0.322$

c Increase the sample size or increase the significance level of the test.

6 a H_0 : the number of people with a car follows a B(50,0.3) distribution

b Need to find smallest a such that $P(X \ge a) \le 0.05$. $P(X \ge 20) = 0.0848 \not\le 0.05$,
$P(X \ge 21) = 0.0478 \le 0.05$ so critical region is $X \ge 21$

c i $P(X < 21 \mid p = 0.4) = 0.561$ **ii** $P(X < 21 \mid p = 0.5) = 0.101$

7 a The hours chosen for the sample need to be independent. If they were adjacent to each other, then an event such as a large car crash could mean that several of the hours had more arrivals than usual.

b $H_0 : \mu = 82$, $H_1 : \mu < 82$

c A type I error would occur if the average number of patients has not fallen but the sample mean ≤75 such that the null hypothesis is rejected. This would mean the phone line is continued unnecessarily. For $X \sim \text{Po}(82)$, $P(X \le 75 \mid \mu = 82) = 0.239$.

d $P(X \ge 76 \mid \mu = 78) = 1 - P(X \le 75 \mid \mu = 78) = 0.605$.

This is the probability of accepting the null hypothesis that the mean has not fallen and discontinuing the phone line even though it has had a positive effect.

8 a $H_0 : \mu = 1.2$, $H_1 : \mu \ne 1.2$ **b** 0.05

c Need to find b and c such that $P(b \le \bar{X} \le c) = 0.95$. when $\mu = 1.2,\ \sigma = \dfrac{0.1}{\sqrt{20}}$

$$P(\bar{X} \le b) = 0.025 \Rightarrow b = 1.1562.\ P(\bar{X} \le c) = 0.975 \Rightarrow c = 1.2438$$

So critical regions are $\bar{X} < 1.1562$ and $\bar{X} > 1.2438$.

$P(1.1562 \le \bar{X} \le 1.2438 \mid \mu = 1.17) = 0.731$

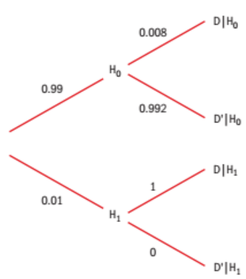
Exercise 14Q

1 a Under the null hypothesis that the shapes are guessed randomly the probability of 3 correct is $0.2^3 = 0.008$

b The probability he is quoting is the probability of not getting all three cards right if guessing randomly, not the probability he is not just guessing randomly.

c How many previous tests had Bruno taken and what were the results? Is there any evidence of trickery or conspiracy between Bruno and the tester? Are the cards marked? Is the distribution of cards in the pack even? Were the cards randomly drawn?

d



e i $P(D) = P(H_0) \times P(D \mid H_0) + P(H_1) \times P(D \mid H_1) = 0.99 \times 0.008 + 0.01 \times 1 = 0.1792$

ii $P(H_1 \mid D) = \dfrac{P(H_1) \times P(D \mid H_1)}{P(D)} = \dfrac{0.01 \times 1}{0.1792} = 0.5580$

f i The researcher's belief that the probability of ESP existing is 0.01.

ii $P(D) = P(H_0) \times P(D \mid H_0) + P(H_1) \times P(D \mid H_1) = 0.999 \times 0.008 + 0.001 \times 1 = 0.008992$

$P(H_1 \mid D) = \dfrac{P(H_1) \times P(D \mid H_1)}{P(D)} = \dfrac{0.001 \times 1}{0.008992} = 0.111$

2 a i $X \sim \mathrm{B}(100, 0.01)$,

$P \quad X \quad 3 \quad 1 \quad P \quad X \quad 2 \quad 0.0794$

ii No, more information is needed.

b i $P(X \ge 3) = P(H_0) \times P(X \ge 3 \mid H_0) + P(H_1) \times P(X \ge 3 \mid H_1)$,

$= 0.9 \times 0.0794 + 0.1 \times 1 = 0.171$

$P(H_1 \mid X \ge 3) = \dfrac{P(H_1) \times P(X \ge 3 \mid H_1)}{P(X \ge 3)} = \dfrac{0.1 \times 1}{0.171} = 0.583$

ii The probability of bad practice is just under 60%. Though not significant, further checks might be advisable.

3 a Need to find a such that $P(\bar{X} \le a) = 0.05$, given $\mu = 5.2, \sigma = \frac{1.2}{\sqrt{16}}$

$P(\bar{X} \le a) \Rightarrow a = 4.7065$. So critical region is $\bar{X} < 4.71$.

b $P(\bar{X} \ge 4.7065 \mid \mu = 4.6) = 0.361$

c $P(X \le 4.7065) = P(\text{inside}) \times P(X \le 4.7065 \mid \text{inside}) + P(\text{outside}) \times P(X \le 4.7065 \mid \text{outside})$
$= 0.9 \times 0.05 + 0.1 \times (1 - 0.361) = 0.109$

d $P(\text{inside} \mid X \le 4.7065) = \dfrac{P(\text{inside}) \times P(X \le 4.7065 \mid \text{inside})}{P(X \le 4.7065)} = \dfrac{0.9 \times 0.05}{0.109} = 0.413$

Chapter review

1 a

x	y	R_x	R_y
151	17.5	1	11
153	18	2.5	12
153	16.5	2.5	10
154	16	4	9
155	15.4	5	8
159	13.2	6	4.5
162	14	7	7
164	13.7	8.5	6
164	13.2	8.5	4.5
168	12.5	10	3
175	12	11	1.5
181	12	12	1.5

$r_s = -0.9525$

b The correlation is strong and negative, so, the taller the person, the less time they take to run the 100 metres. The sample is fairly small. The time at which someone runs the 100m will also depend on factors such as their age, gender and physical fitness.

2 a H_0 : egg colour is independent of type of hen

H_1 : egg colour isn't independent of type of hen

b $\frac{30}{90} \times \frac{42}{90} \times 90 = 14$

c Degrees of freedom $= (3 - 1) \times (2 - 1) = 2 \times 1 = 2$

d

Expected	Leghorn	Brahma	Sussex

White eggs	14	14	14
Brown eggs	16	16	16

$\chi^2 = 21.7$

$p\text{-value} = 0.0000194$

e $21.7 > 5.991$ and $0.0000194 < 0.05$. Significant so reject the null hypothesis. The colour of the eggs is not independent of the type of hen.

3 **a** $P(0\text{ tails}) = \frac{1}{2} \times \frac{1}{2} = 0.25$ so expected frequency for tossing 0 tails is

$60 \times 0.25 = 15$

b

Number of tails	**0**	**1**	**2**
Probability	0.25	0.5	0.25
Expected Frequency	15	30	15

c 2 degrees of freedom

d H_0 : the data fits a binomial distribution

H_1 : the data doesn't fit a binomial distribution

$\chi 2 = 1.2$ or p-value $= 0.549$

e 1.2< 5.991 or 0.549 > 0.05, so not significant and no reason to reject the null hypothesis that the data fits a binomial distribution.

4 **a** $H_0 : \mu_1 = \mu_2$, $H_1 : \mu_1 \neq \mu_2$

b Two tailed

c p-value = 0.678

0.678 > 0.05, not significant so no reason to reject the null hypothesis that there is no difference between the two groups.

5 Using the confidence interval function on the GDC

a $(1.3733, 1.741)$

b (21.611,22.589)

6 $H_0 : \lambda = 15.0$, $H_1 : \lambda > 15.0$, assume H_0 is true, then $P(X \geq 19) = 0.1805 > 0.05$, not significant so insufficient evidence to reject H_0 that the number of hurricanes is still 7.5 on average.

7 **a** $X \sim \mathrm{B}(42, 0.82)$

There are just two possible outcomes and the recovery of patients is likely to be independent of all other patients.

b $H_0 : p = 0.82$, $H_1 : p < 0.82$

c Need to find r such that $P(X \leq r) = 0.05$. $r = 29 \Rightarrow P(X \leq r) = 0.0293$,

$r = 30 \Rightarrow P(X \leq r) = 0.0625$ so the critical region is $X \leq 29$

d 0.0293

8 **a** **i** 0.05 **ii** 0.01

b **i** Need to find r such that $P(\bar{X} \geq r) = 0.05$ when $\mu = 30,\ \sigma = \frac{3}{\sqrt{4}}$

$P(\bar{X} \leq r) = 0.95 \Rightarrow r = 32.4674$, so the critical region is $\bar{X} \geq 32.47$ and
$P(\bar{X} < 32.4674 \mid \mu = 32) = 0.622$

ii Need to find r such that $P(\bar{X} \geq r) = 0.01$.

$P(\bar{X} \leq r) = 0.99 \Rightarrow r = 33.4895$, so the critical region is $\bar{X} \geq 33.49$ and
$P(\bar{X} < 33.4895 \mid \mu = 32) = 0.840$

c **i** Decrease **ii** Increase **iii** Decrease

9 **a** Test-retest means the same test is given to the same people after a period of time. If the test is reliable there should be a good correlation between the two sets of results.

b

$r = 0.968$

This indicates that the test is very reliable.

c Let μ_D be average difference between the scores on the first and second tests.

Differences	0.2	0.4	0.4	0	−0.5	0.6	−0.3	0.6

$H_0 : \mu_D = 0$, $H_1 : \mu_D \neq 0$

p-value $= 0.133 > 0.05$, not significant at 10% so no reason to reject H_0 that there has been no increase in the overall level of satisfaction.
Assumptions: The differences can be modeled by a normal distribution and the responses of those surveyed were independent of each other.

Exam-style questions

10 Let X be the number of heads.

a Probability of getting 4 heads with a fair coin is $P(X = 4 \mid p = 0.5) = \left(\frac{1}{2}\right)^4 = 0.0625$.

b 0

11 **a** Use a chi-squared test for independence
H_0 : favourite TV channel is independent of age
H_1 : favourite TV channel isn't independent of age

Expected	**Alpha**	**Beta**	**Peppa**
Up to 5 years old	12	21	27
Between 6 and 10 years	14	24.5	31.5

Between 11 and 15 years	14	24.5	31.5

Degrees of freedom $= (3-1)\times(3-1) = 2\times 2 = 4$

p-value $= 0.000279$

$0.000279 < 0.01$. Significant so reject the null hypothesis. The favourite TV channel is not independent of age.

b Table above: values are valid as they are all >5

12 a i Welsh mean $\bar{x}_1 = 173.75$

ii Scottish mean $\bar{x}_2 = 177.8$

b 2-sample t-test since variance is unknown.

H_0: $\mu_{Welsh} = \mu_{Scottish}$

H_1: $\mu_{Welsh} < \mu_{Scottish}$

p-value $= 0.0214 < 0.05$, so we reject the null hypothesis and conclude that there is sufficient evidence at the 5% level to conclude that Welsh policemen are shorter than Scottish policemen.

c 0.0214 > 0.01 so at the 1%, level we would accept H_0.

13 One tailed test. $H_0 : \lambda = 20.0$, $H_1 : \lambda < 20.0$.

Here $X \sim \text{Po}(20\times 6) = \text{Po}(120)$.

Assume H_0 is true, then $P(X \le 100) = 0.0347 < 0.05$, significant so sufficient evidence to reject H_0, suggesting that Narcissus is exaggerating.

14 a If X has a mean of μ and a standard deviation of σ then the mean of a sample that is sufficiently large (>30), has distribution $\overline{X} \sim \text{N}\left(\mu, \frac{\sigma^2}{n}\right)$.

b Using the inbuilt function on the GDC for a confidence interval when the variance is known (Z) (4.804,5.196)

15 A paired t test is used because we wish to directly compare measurements associated with the same cars. Let μ_D be average difference between the scores on the first and second tests.

Differences	0.2	0	0.4	−0.1	0.6	0	0.2	0.1	−0.1	0.3

$H_0 : \mu_D = 0$, $H_1 : \mu_D > 0$

p-value $= 0.0264 < 0.05$, significant at 5% so evidence to reject H_0 that front wheels wear at the same rate as the rear wheels.

16 H_0: there is not a linear relationship, $(\rho = 0)$ H_1: there is a linear relationship, $(\rho \neq 0)$

p-value is $0.7996 > 0.05$ so not significant, so not enough evidence to reject the null hypothesis that there is no linear relationship between a female's height and the number of hours spent on social media.

17 Use a Chi squared goodness of fit test.

H_0 : toys appear in the colour ratio 3:4:2:1

H_1 : toys don't appear in the distribution stated

Degrees of freedom is 4-1=3

Colour	**Blue**	**Pink**	**Purple**	**Green**
Observed	32	37	23	8
Expected	30	40	20	10

p-value $= 0.7510$

$0.7510 > 0.05$. Not significant so do not reject the null hypothesis. The colour of the toys follows the stated distribution.

15 Optimizing complex networks: graph theory

Skills check

1 a $\begin{pmatrix} -1 & -2 & -1 \\ 3 & -1 & 3 \\ -3 & 1 & 4 \end{pmatrix}$ **b** $\begin{pmatrix} -2 & 3 & -9 \\ -15 & -2 & 6 \\ -6 & 9 & 22 \end{pmatrix}$

2 a Let $P = \begin{pmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{pmatrix}$. Form equations $0.8x + 0.3y = x$ and $0.2x + 0.7y = y$, which simplify to $3y = 2x$. Solving either of those with $x + y = 1$ gives $x = 0.6$ and $y = 0.4$.

The steady state probability vector is $\begin{pmatrix} 0.6 \\ 0.4 \end{pmatrix}$

b For instance, using technology we find that $P^{10} = \begin{pmatrix} 0.6004 & 0.5994 \\ 0.3996 & 0.4006 \end{pmatrix}$.

c The long-term probability of being in state 1 is 0.6 and in state 2 is 0.4.

Exercise 15A

1 All except **c**, because there is no edge AE.

2 a

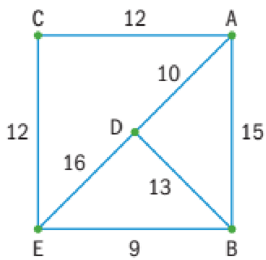
	A	B	C	D	E
A		1	1	1	
B	1		1		
C	1	1		1	
D	1		1		1
E				1	

b

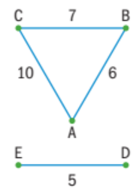
	P	Q	R	S	T
P		1	1		
Q	1		1	1	1
R	1	1			
S		1			2
T		1		2	

3

a

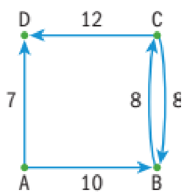


b

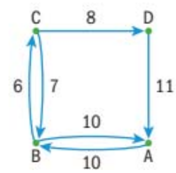


4

a

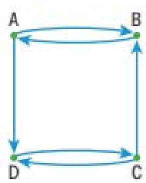


b

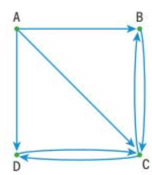


5

a



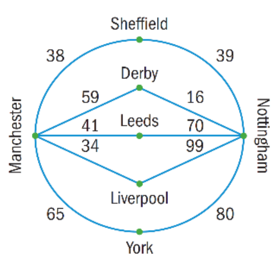
b



6 **a** Sheffield to Manchester is 38 miles and Nottingham to Sheffield is 39 miles.

b One possible route is Manchester to Sheffield to Nottingham, which is 77 miles. A route via any town more than 77 miles from Manchester will necessarily be longer.

c



d The shortest distance between Manchester and Nottingham through one other town is 75 miles through Derby. Therefore, the shortest distance can be estimated as $38 + 75 + 39 = 152$ miles.

Exercise 15B

1 Cases **a** and **c** are undirected, **b** and **d** are directed.

2 a i

	A	B	C	D
A		1		
B	1			1
C	1			1
D			1	

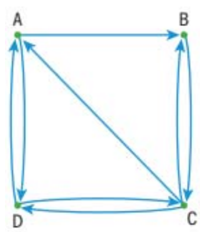
ii

	A	B	C	D
A				
B	8			
C	9			4
D		7	5	

b i Strongly connected

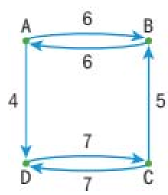
ii Connected but not strongly connected

3 a i



ii A has in-degree 2 and out-degree 2, B has in-degree 2 and out-degree 1, C has in-degree 2 and out-degree 3, and D has in-degree 2 and out-degree 2.

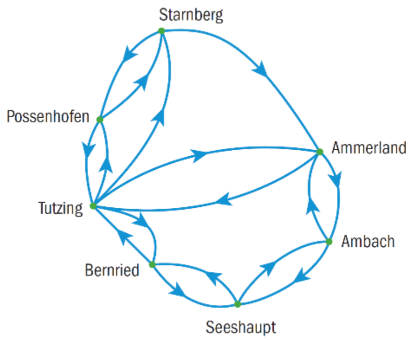
b i



ii A has in-degree 1 and out-degree 2, B has in-degree 2 and out-degree 1, C has in-degree 1 and out-degree 2, and D has in-degree 2 and out-degree 1.

4 A has in-degree 1 and out-degree 3, B has in-degree 2 and out-degree 1, C has in-degree 2 and out-degree 2, D has in-degree 2 and out-degree 2, and E has in-degree 3 and out-degree 2.

5 **a**



b Any one from: It is easier to see which towns are connected, the routes between the towns, and which journeys have a return voyage.

c Starnberg and Ammerland, and Tutzing and Starnberg

d Starnberg, Ammerland, Ambach; Starnberg, Ammerland, Tutzing, Bernried, Seeshaupt, Ambach; Starnberg, Possenhofen, Tutzing, Ammerland, Ambach; Starnberg, Possenhofen, Tutzing, Bernried, Seeshaupt, Ambach.

e All of them except Starnberg, Ammerland, Tutzing, Bernried, Seeshaupt, Ambach. Starnberg-Ammerland arrives at 12.44 at the earliest, and Ammerland-Tutzing leaves at 11.51.

Exercise 15C

1 **a**

$$\begin{array}{cc} & \begin{matrix} A & B & C & D \end{matrix} \\ \begin{matrix} A \\ B \\ C \\ D \end{matrix} & \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \end{array}$$

b

$$\begin{array}{cc} & \begin{matrix} A & B & C & D \end{matrix} \\ \begin{matrix} A \\ B \\ C \\ D \end{matrix} & \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

c

$$\begin{array}{cc} & \begin{matrix} A & B & C & D & E \end{matrix} \\ \begin{matrix} A \\ B \\ C \\ D \\ E \end{matrix} & \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

d

$$\begin{array}{cc} & \begin{matrix} A & B & C & D & E & F \end{matrix} \\ \begin{matrix} A \\ B \\ C \\ D \\ E \\ F \end{matrix} & \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

2 The adjacency matrix consists only of zeros and ones and has zeros on the diagonal.

3 **a** Sum of entries in either row or column headed by that vertex

b **i** Sum of elements in the column headed by that vertex

ii Sum of elements in the row headed by that vertex

4 **a** **i** Everyone knows their own name

ii It is possible for someone to know the name of someone else without them knowing their name

b **i** 3 **ii** 3 **iii** C **iv** E

c D

Exercise 15D

1 **a** **i** Calculate M^3 and find the element for i in the first row-first column = $M^3{}_{11} = 2$

From the graph the walks are ABDA and ADBA.

ii $M^3{}_{34} = 5$, and from the graph the walks are CBAD, CDBD, CDAD, CDCD, CBCD.

b **i** 1; ABDA **ii** 0

c **i** 2; ABEA, AEBA **ii** 5; CBED, CBCD, CDCD, CDBD, CDED

d **i** 0 **ii** 1; CCCD

2 $S = M + M^2 = \begin{pmatrix} 3 & 3 & 1 & 1 & 3 \\ 3 & 4 & 3 & 1 & 2 \\ 1 & 3 & 4 & 4 & 4 \\ 0 & 1 & 3 & 4 & 4 \\ 0 & 0 & 1 & 3 & 3 \end{pmatrix}$

Since $S_{41} = 0$, there are no walks of length 1 or 2 connecting D to A.

3 **a** **i** $M^2 = \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 3 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}$, $M^3 = \begin{pmatrix} 2 & 3 & 4 & 1 \\ 3 & 2 & 4 & 1 \\ 4 & 4 & 2 & 3 \\ 1 & 1 & 3 & 0 \end{pmatrix}$

ii This is given by the element $M^2{}_{31} = 1$.

iii $S_3 = M + M^2 + M^3 = \begin{pmatrix} 4 & 5 & 6 & 2 \\ 5 & 4 & 6 & 2 \\ 6 & 6 & 5 & 4 \\ 2 & 2 & 4 & 1 \end{pmatrix}$, therefore 6.

iv M has zeroes other than along the leading diagonal. For example, there is no path of length 1 between A and D. Hence, the diameter must be greater than 1.

$S_2 = M + M^2 = \begin{pmatrix} 2 & 2 & 2 & 1 \\ 2 & 2 & 2 & 1 \\ 2 & 2 & 3 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$

There are no zeros in S_2, so any vertex can be reached from any other vertex in a path of length 1 or 2. Thus, the diameter is 2.

b i

$$M^2 = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 & 1 \\ 1 & 0 & 3 & 1 & 1 \\ 0 & 1 & 1 & 2 & 1 \\ 0 & 1 & 1 & 1 & 2 \end{pmatrix}, \quad M^3 = \begin{pmatrix} 0 & 2 & 0 & 1 & 1 \\ 2 & 0 & 4 & 1 & 1 \\ 0 & 4 & 2 & 4 & 4 \\ 1 & 1 & 4 & 2 & 3 \\ 1 & 1 & 4 & 3 & 2 \end{pmatrix}$$

ii This is given by the element $M^2{}_{31} = 1$.

iii $S_3 = M + M^2 + M^3 = \begin{pmatrix} 1 & 3 & 1 & 1 & 1 \\ 3 & 2 & 5 & 2 & 2 \\ 1 & 5 & 5 & 6 & 6 \\ 1 & 2 & 6 & 4 & 5 \\ 1 & 2 & 6 & 5 & 4 \end{pmatrix}$, therefore 1.

iv $S_1 = M$ and $S_2 = M + M^2 = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 1 & 1 \\ 1 & 1 & 3 & 2 & 2 \\ 0 & 1 & 2 & 2 & 2 \\ 0 & 1 & 2 & 2 & 2 \end{pmatrix}$ contain zeros but S_3 does not, so diameter is 3

4 a The number of routes between Lochboisdale and Castlebay in three ferry trips is given by the element $M^3{}_{31}$. The third row of M^2 is $(1, 0, 2, 0, 0, 2, 1)$, so $M^3{}_{31} = (1, 0, 2, 0, 0, 2, 1) \cdot (0, 0, 1, 0, 0, 1, 1) = 2 + 2 + 1 = 5$.

b The sixth row of M is $(1, 0, 0, 0, 0, 0, 1)$, so it is possible to reach Castlebay and Tobermory in one ferry trip. The sixth row of M^2 is $(1, 0, 2, 0, 0, 2, 1)$, so it is possible to reach Lochboisdale in two trips. The sixth row of M^3 is $(5, 0, 2, 0, 0, 2, 5)$, and we can see from the structure of M that for any higher powers there will be no new routes. Thus, you can get to Castlebay, Lochboisdale and Tobermory from Tiree.

c $M = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 \end{pmatrix}$.

$S_1 = M$ contains zeros and $S_2 = M + M^2$ also contains zeros. Checking higher powers, we find that the first matrix S_r that does not contain zeros is

$$S_5 = \begin{pmatrix} 57 & 2 & 56 & 20 & 10 & 47 & 58 \\ 2 & 3 & 6 & 4 & 8 & 2 & 2 \\ 56 & 6 & 43 & 28 & 8 & 36 & 56 \\ 20 & 4 & 28 & 11 & 14 & 18 & 20 \\ 10 & 8 & 8 & 14 & 7 & 4 & 10 \\ 47 & 2 & 36 & 18 & 4 & 32 & 47 \\ 58 & 2 & 56 & 20 & 10 & 47 & 57 \end{pmatrix}.$$

Therefore, at least one journey between two ports requires 5 ferry trips. Since S_4 has zero elements only for trips between Eigg and Tiree, this route takes 5 trips, and these two ports are the furthest apart.

5 a

$$M = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

, where the town order is Starnberg, Possenhofen, Tutzing, Ammerland, Bernried, Ambach, Seeshaupt.

b This is given by $S_2 = M + M^2$, $M^2 = \begin{pmatrix} 1 & 0 & 2 & 0 & 0 & 1 & 0 \\ 1 & 2 & 0 & 2 & 1 & 0 & 0 \\ 1 & 1 & 3 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 2 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 2 \end{pmatrix}$ and

$$S_2 = \begin{pmatrix} 1 & 1 & 2 & 1 & 0 & 1 & 0 \\ 2 & 2 & 1 & 2 & 1 & 0 & 0 \\ 2 & 2 & 3 & 2 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 2 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 2 \end{pmatrix}$$

, so the towns that are not connected directly or at most one stop are Starnberg to Bernried, Starnberg to Seeshaupt, Possenhofen to Ambach, Possenhofen to Seeshaupt, Ambach to Starnberg, Ambach to Possenhofen, Seeshaupt to Starnberg, Seeshaupt to Possenhofen.

Exercise 15E

1 a The transition matrix P is $\begin{pmatrix} 0 & \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{3} \\ 0 & \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{2} & 0 \end{pmatrix}$

i For instance, $P^{20} = \begin{pmatrix} 0.200 & 0.200 & 0.200 & 0.200 \\ 0.300 & 0.300 & 0.300 & 0.300 \\ 0.200 & 0.200 & 0.200 & 0.200 \\ 0.300 & 0.300 & 0.300 & 0.300 \end{pmatrix}$, therefore the steady state probabilities are $\begin{pmatrix} 0.2 \\ 0.3 \\ 0.2 \\ 0.3 \end{pmatrix}$.

ii Solving $\begin{pmatrix} 0 & \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{3} \\ 0 & \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{2} & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$ gives $-3x_1 + x_2 + x_4 = 0$, $3x_1 - 6x_2 + 3x_3 + 2x_4 = 0$, $x_2 - 3x_3 + x_4 = 0$ and $3x_1 + 2x_2 + 3x_3 - 6x_4 = 0$. Normalization requires that all probabilities add up to one: $x_1 + x_2 + x_3 + x_4 = 1$.

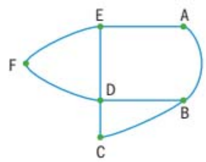
Solving this system gives $x_1 = 0.2, x_2 = 0.3, x_3 = 0.2, x_4 = 0.3$, and therefore $x = \begin{pmatrix} 0.2 \\ 0.3 \\ 0.2 \\ 0.3 \end{pmatrix}$.

b The transition matrix P is $\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & 0 \end{pmatrix}$

i For instance, $p^{25} = \begin{pmatrix} 0.462 & 0.462 & 0.462 & 0.462 \\ 0.308 & 0.308 & 0.308 & 0.308 \\ 0.0769 & 0.0769 & 0.0769 & 0.0769 \\ 0.154 & 0.154 & 0.154 & 0.154 \end{pmatrix}$, and there is no change in the third significant figure at higher powers. Therefore, the steady-state probabilities are $\begin{pmatrix} 0.462 \\ 0.308 \\ 0.0769 \\ 0.154 \end{pmatrix}$.

ii The system of equations is $-x_1 + x_2 + x_4 = 0$, $x_1 - 2x_2 + 2x_3 = 0$, $-2x_3 + x_4 = 0$, $x_2 - 2x_4 = 0$ and $x_1 + x_2 + x_3 + x_4 = 1$. This gives $x = \begin{pmatrix} 6/13 \\ 4/13 \\ 1/13 \\ 2/13 \end{pmatrix} \approx \begin{pmatrix} 0.462 \\ 0.308 \\ 0.0769 \\ 0.154 \end{pmatrix}$.

2 a



b The transition matrix P is $\begin{pmatrix} 0 & \frac{1}{3} & 0 & 0 & \frac{1}{3} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{3} & 0 & \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{2} & 0 & \frac{1}{3} & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & \frac{1}{4} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{4} & \frac{1}{3} & 0 \end{pmatrix}$

c $P^4 = \begin{pmatrix} \frac{2}{9} & \frac{1}{18} & \frac{11}{72} & \frac{11}{72} & \frac{1}{18} & \frac{11}{72} \\ \frac{1}{12} & \frac{5}{18} & \frac{5}{32} & \frac{5}{32} & \frac{1}{4} & \frac{5}{32} \\ \frac{11}{72} & \frac{5}{48} & \frac{47}{288} & \frac{17}{144} & \frac{5}{48} & \frac{13}{96} \\ \frac{11}{36} & \frac{5}{24} & \frac{17}{72} & \frac{43}{144} & \frac{5}{24} & \frac{17}{72} \\ \frac{1}{12} & \frac{1}{4} & \frac{5}{32} & \frac{5}{32} & \frac{5}{18} & \frac{5}{32} \\ \frac{11}{72} & \frac{5}{48} & \frac{13}{96} & \frac{17}{144} & \frac{5}{48} & \frac{47}{288} \end{pmatrix}$, therefore the probability of being in room A after four passages is $\frac{2}{9} \approx 0.222$

d i We can find the long-term probability by considering high powers of the matrix P, for instance by using technology. We find that the first column of P^{25} is given by $(0.125, 0.188, 0.125, 0.250, 0.188, 0.125)$ (higher powers only change at most the fifth digit). Therefore, the rooms that are most likely to be visited by the robot are D, B and E.

ii The percentage of time is 25% in D and 18.8% in each of B and E.

3 a The transition matrix T is given by $\begin{pmatrix} 0 & 0.5 & 0.5 & 0 \\ 1 & 0 & 0.5 & 0.5 \\ 0 & 0 & 0 & 0.5 \\ 0 & 0.5 & 0 & 0 \end{pmatrix}$

b $T^3 = \begin{pmatrix} 0 & 0.5 & 0.25 & 0.125 \\ 0.75 & 0.125 & 0.375 & 0.625 \\ 0.25 & 0 & 0.125 & 0.125 \\ 0 & 0.375 & 0.25 & 0.125 \end{pmatrix}$

Hence, in 3 steps it not possible to go from B to C, or from A to A and A to D.

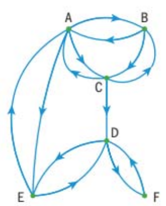
c **i** Solving $Tx = x$ gives the system of equations $-x_1 + 0.5x_2 + 0.5x_3 = 0$, $x_1 - x_2 + 0.5x_3 + 0.5x_4 = 0$, $-x_3 + 0.5x_4 = 0$, $0.5x_2 - x_4 = 0$ together with $x_1 + x_2 + x_3 + x_4 = 1$. Solving this system, we get $x_1 = \frac{5}{19}$, $x_2 = \frac{8}{19}$, $x_3 = \frac{2}{19}$, $x_4 = \frac{4}{19}$.

The steady state probabilities are $\left(\frac{5}{19}, \frac{8}{19}, \frac{2}{19}, \frac{4}{19}\right)$.

ii The proportion of time that a person will spend on site B will be $\frac{8}{19}$.

d B, A, D, C

4 **a**



b It would be possible only if George ended on Frances's page, otherwise he would need to visit Dawn's page twice. This is not possible as you cannot pass through all the other pages and end at Dawn's page without repeating some pages.

c Yes, if you begin on Frances's page: Frances, Dawn, Emil, Antoine, Belle and Charles

d $\begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{3} & 0 & \frac{1}{2} & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & \frac{1}{2} & 1 \\ \frac{1}{3} & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \end{pmatrix}$

e The first row of the fifth power of the transition matrix is $\left(\frac{85}{648}, \frac{293}{1296}, \frac{1051}{3888}, \frac{1}{36}, \frac{793}{2592}, \frac{43}{144}\right)$. Therefore if the initial state is $(1, 0, 0, 0, 0, 0)$

the probability that after five visits he is back on Antoine's page is $\frac{85}{648} \approx 0.131$.

f The total probability is given by the product of individual probabilities of each step

$\frac{1}{3}\times\frac{1}{2}\times\frac{1}{3}\times\frac{1}{2}\times\frac{1}{2}=\frac{1}{72}$

g Using technology, the first column of P^{130} is $\begin{pmatrix}0.185\\0.0988\\0.111\\0.272\\0.198\\0.136\end{pmatrix}$,

where the power is such that the variation is at the fourth digit at most. If he starts on Antoine's page, these are the long-term probabilities.

i He is most likely to visit Dawn's page. **ii** He is least likely to visit Belle's page.

5 **a** Because two states are absorbing states and the probability of being in them will depend on the starting position.

b Because A is absorbing state, and it is impossible to get from B to E, the probability that a random walk would end at vertex A if it began at B is 1.

c It is possible to reach A from B only either directly with the walk BA with probability 0.5 or by going through the vertex F, so other possible paths are BFBA, BFBFBA, BFBFBFBA, ... Therefore, using technology,, the expectation value is

$$\begin{aligned}E(X)&=0.5\times1+0.5^2\times3+0.5^3\times5+\ldots\\&=\sum_{i=1}^{\infty}0.5^i(2i-1)\\&=3.\end{aligned}$$

d $\begin{pmatrix}1&\frac{1}{2}&0&0&0&0\\0&0&\frac{1}{3}&0&0&1\\0&0&0&\frac{1}{3}&0&0\\0&0&\frac{1}{3}&0&0&0\\0&0&0&\frac{1}{3}&1&0\\0&\frac{1}{2}&\frac{1}{3}&\frac{1}{3}&0&0\end{pmatrix}$

e We find $P^{30}=\begin{pmatrix}1&1&0.875&0.625&0&1\\0&0.000&0.000&0.000&0&0\\0&0&0.000&0&0&0\\0&0&0&0.000&0&0\\0&0&0.125&0.375&1&0\\0&0&0.000&0.000&0&0.000\end{pmatrix}$, and if the initial state is $(0,0,1,0,0,0)$,

the long term probabilities are given by $(0.875,\ 0.000,\ 0.000,\ 0,\ 0.125\ ,0.000)$, and so the probability of being at vertex E is 0.125.

Exercise 15F

1 **a** Using Prim's algorithm and beginning with vertex A, we select edges in the order of AF, FE, FD, DC, AB. The minimum weight is $8+5+6+7+9=35$.

b We select edges AB, AF, FE, ED, DC, DG, with the weight $10+11+13+14+9+10=67$.

c We select edges AF, FC, CB, CD, FE, with the weight $4+5+3+5+6=23$.

d We select edges AB, BF, FE, BG, FC, CD or AB, BF, FE, FC, CD, BG, both with weight $8+6+5+7+7+6=39$.

2 **a** The lowest weight edge is EF. Starting with this edge, we select EF, FD, DC, FA, AB with weight $5+6+7+8+9=35$.

b We select CD, AB (or GD), GD (or AB), AF, FE, ED with the weight $9+10+10+11+13+14=67$.

c We select BC, AF, FC (or CD), CD (or FC), FE with the weight $3+4+5+5+6=23$.

d We select EF, CD (or FB), FB (or CD), CF (or BG), BG (or CF), AB with the weight $5+6+6+7+7+8=39$.

3 You begin with a single vertex. Every time you add a new vertex to the tree you also add an edge. You need to add $v-1$ vertices and so the spanning tree will have $v-1$ edges.

4 **a** We select AE (the lowest weight), CD, ED (or CB), CB (or ED), AF, with the weight $6+7+8+8+9=38$. Therefore, the minimum cost is $38,000.

b **i** Begin by connecting B and D and apply the algorithm from this point.

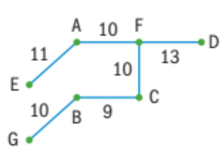
ii We select BD, AE, DC, DE, AF, with the weight $10+6+7+8+9=40$. The cost in this case is $40,000.

5 **a** Starting with vertex A, we select AD, DC, AE, CB, with the weight $15+15+20+25=75$.

b Starting with vertex A, we select AB, BE, EC, BD or AB, BE, BD, EC, both with weight $4+3+5+5=17$.

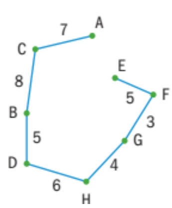
c Because it is difficult to check whether adding an edge will form a cycle.

6 **a** Using Prim's algorithm and starting from vertex A, we select AF, FC, CB, BG, EA, DF $10+10+9+10+11+13=63$



b Using Prim's algorithm with edge AB, we get AB, BC, CF, BG, AE, FD, or AB, BC, AF, BG, AE, FD or AB, BC, BG, AF or CF, AE, FD, all giving $20+9+10+10+11+13=73$. Therefore, the extra length of pipe that will be required is 10 m.

7 **a** **i** We select AC, CB, BD, DH, HG, GF, FE.



ii The weight is $7+8+5+6+4+3+5=38$.

b **i** We select AC, CB, BD, DH, HG, GF, FE, IA with the weight $38+9=47$.

ii We select AI, AC, IB, BD, DH, HG, GF, FE, with the weight $5+7+6+5+6+4+3+5=41$.

Exercise 15G

1 **a** There are two odd vertices, C and E. The route of least weight between these vertices is CFE. A route is ABFCFEFGBCDEGA, with repeated edges CF and EF and weight 103. This is the total weight plus the weight of the repeated edge, $91 + 7 + 5 = 103$.

b There are two odd vertices, F and A with the route of least weight between them ABF. A route is ABCDEFBFDBA, with repeated edges AB and BF and weight $(20+20+30+40+35+50+45+45)+35+20=340$.

c There are two odd vertices, F and C with the route of least weight between them FBC (or equivalently FEC). A route is ABCDEFBECEFA with repeated edges CE and EF or ABCDEFBECBFA with repeated edges CB and BF. Both Eulerian circuits have weight $(5+4+7+7+9+6+6+4+5)+6+7=66$.

2 **a** There are two vertices of odd degree, E and G. An Eulerian circuit requires that all the vertices have even degree. For there to be an Eulerian trail, there must be exactly two odd vertices.

b **i** The route of least weight between the two odd vertices G and E is GFE, therefore the solution is $(13+9+12+17+14+23+12+16+14+14+16)+\frac{1}{3}(14+16)$

$=160+10=170$ min.

He would need to walk along GF and FE twice.

ii If every edge is duplicated then every vertex would have an even degree. An Eulerian circuit would be any route that traverses each edge of the graph twice, for instance, ABCDEFGABGBFBCFCECDEFGA, or any Eulerian trail immediately followed by the same trail.

3 **a** There are two odd vertices, G and E, and the cheapest route between them is GBE. Therefore, the minimum possible cost is $(5+6+7+6+3+3+4+5+5+6)+4+5=59$ dollars.

b The cost will go down as all vertices now have even degree and so no routes need be repeated. There is an additional cost of \$7 but a saving of \$9. Total cost is \$57.

Exercise 15H

1 **a** Odd vertices are H, B, F, D. The pairings are HB (4) and FD (5), HF (6) and BD (5), and HD (6) and BF (7). The pair with the minimum weight is HB (4) and FD (5). Therefore, the edges that need to be repeated are HB, DI and IF.

b Odd vertices are E, F, B, C. The pairings are BC (8) and FE (11), BF (15) and CE (13), and BE (21) and CF (7). The pair with the minimum weight is BC (8) and FE (11), and so repeated edges are BC and FE.

2 **a** The odd vertices are I, F, D, C. The pairings are IHF (100) and DGC (70), IHGD (110) and CGF (90), and IBC (110) and DEF (70). Therefore, the lowest weight pairing is IF and DC, and the repeated edges are IH, HF, DG and GC. One possible route is

AIHFHIBHGFEDGCDGCBA, with length $650+170=820$. Here 650 is the sum of weights of all edges in the graph, and 170 is the weight of the repeated edges.

b Here we will produce an Eulerian trail rather than a circuit, so that the start and end vertices only have odd degree. The two pairs with the lowest weights are DC (70) and DF (70), so the routes between these pairs should be repeated. Since they both weigh 70, the overall weight would be 650 + 70 = 720. If we repeat DGC, then we would start/finish at I and F. If we repeat DEF, then we would start/finish at I and C.

3 a Odd vertices are A, B, D and H. The pairings are AB (38) and DGH (49), AGD (43) and BAH (73), and AH (35) and BD (88). So, we need to repeat AB, DG and GH. A possible route is ABCDGABDGFDEFHGHA with weight $526+38+49=613$ metres, where 526 is the sum of weights of all edges in the graph.

b He would have to repeat BD, which is 88 m. Previously he had to repeat AB and DH, which total 87 m.

c As he needs to return to A the best place to be picked up is at B so he will only need to repeat DH, so length of repeated roads will be 49 m.

Exercise 15I

1 a i

	A	**B**	**C**	**D**	**E**
A	0	9	14	8	10
B	9	0	7	4	7
C	14	7	0	6	9
D	8	4	6	0	3
E	10	7	9	3	0

ii Nearest neighbour algorithm gives ADEBCA. Actual route is ADEDBCDA with the weight $8+3+3+4+7+6+8=39$.

b i

	A	**B**	**C**	**D**	**E**	**F**
A	0	6	7	12	10	4
B	6	0	3	8	11	8
C	7	3	0	5	8	5
D	12	8	5	0	7	10
E	10	11	8	7	0	6
F	4	8	5	10	6	0

ii Nearest neighbour algorithm gives AFCBDEA, with the actual route AFCBCDEFA. The weight is $4+5+3+3+5+7+6+4=37$.

c i

	A	**B**	**C**	**D**	**E**	**F**
A	0	6	6	5	4	9
B	6	0	4	6	9	8
C	6	4	0	2	5	4

D	5	6	2	0	3	6
E	4	9	5	3	0	5
F	9	8	4	6	5	0

ii Hamiltonian cycle AEDCBFA. Another possible cycle is AEDCFBA. Actual routes are AEDCBCFEA with weight $4+3+2+4+4+4+5+4=30$ and AEDCFCBA with weight $4+3+2+4+4+4+6=27$. Therefore, the better upper bound is 27.

d i

	A	B	C	D	E
A	0	40	45	25	10
B	40	0	25	15	30
C	45	25	0	20	35
D	25	15	20	0	15
E	10	30	35	15	0

ii Hamiltonian cycle is AEDBCA, with the actual route AEDBCDEA with the weight 110.

2 a Because the graph does not obey the triangle inequality, the upper bound produced by the nearest neighbour algorithm is far higher than the solution to the travelling salesman problem. The actual solution (starting from vertex A) is ACDBA of weight $5+4+6+3=18$. The algorithm gives ABCDA, and includes AD, so has a much higher weight.

b Working directly from the graph without using a table of least distances, we would have ACDFE... or ADCB... From here, neither walk can continue without passing through a vertex that has already been reached.

Exercise 15J

1 a i The minimal spanning tree is BD, DE, DC with weight $4+3+6=13$. So the lower bound on adding edges AB and AD is $13+9+8=30$.

ii No other lower bound is higher. For example, delete D. Then the minimal spanning tree is AB, BE, BC with weight $9+7+7=23$. Adding the edges DB and DE, the lower bound is $23+4+3=30$.

iii The solution to the travelling salesman problem is between 30 and 39.

b i The minimal spanning tree is BC, CF, CD, FE with weight $3+5+5+6=19$. So the lower bound adding edges AB and AF is $19+6+4=29$.

ii Delete C. The minimal spanning tree is AF, AB, FE, ED with weight $4+6+6+7=23$. So the lower bound adding edges CB and CD is $23+3+5=31$. Deleting B gives a lower bound of 29 and deleting D or E gives 30.

iii The solution of the TSP is between 30 or 31 (depending on which vertex you choose) and 37.

c **i** The minimal spanning tree is BC, CD, DE, FC with weight $4+2+3+4=13$. So the lower bound adding edges AD and AE is $13+5+4=22$.

ii Delete C. The minimal spanning tree is AE, ED, EF, AB with weight $4+3+5+6=18$. So the lower bound adding edges CB and CD is $18+4+2=24$. Deleting B gives 23 and deleting D, E or F gives 22.

iii The solution of the TSP is between 23 or 24 (depending on which vertex you choose) and 27.

d **i** The minimal spanning tree is BD, DE, DC with weight $15+15+20=50$. So the lower bound adding edges AD and AE is $50+25+10=85$.

ii Delete D. The minimal spanning tree is AE, EB, BC with weight $10+30+25=65$. So the lower bound adding edges DB and DE is $65+15+15=95$. Deleting B, C or E gives 85.

iii The solution of the TSP is between 95 and 110.

2 **a** Deleting the vertex E, the minimal spanning tree is ADCB of weight $25+40+30=95$. To find the lower bound we add two edges of lowest weight that connect to E, which are AE and EC, so the lower bound is $95+10+10=115$.

b

	A	**B**	**C**	**D**	**E**
A	0	35	20	25	10
B	35	0	25	35	15
C	20	25	0	30	10
D	25	35	30	0	20
E	10	15	10	20	0

c The Hamiltonian cycle is AECBDA, and the actual route is AECEBEDA with weight $10+10+10+15+15+20+25=105$. The upper bound is 105.

d This is lower than the lower bound, because the shortest routes between adjacent vertices is not always the direct route (the triangle inequality does not hold on the graph).

3 Using Prim's algorithm, the minimum spanning tree with vertex A deleted is BC, CD, CG, GE, GF with weight $9+7+6+5+10=37$. We need to add the weight of two lowest weight edges connected to A, AF and AG. Therefore, the lower bound is $37+5+10=52$ hours.

4 **a** Note that this is a complete graph. A route is ADBCEA, with the weight $3+3+4+8+5=23$ minutes.

b Let us delete vertex A. In that case the minimum spanning tree is EDBC with weight $6+3+4=13$. Therefore, the lower bound is $13+3+5=21$.

Alternatively, delete E. The minimum spanning tree is ADBC with weight 3 + 3 + 4 = 10, adding on 5 + 6 = 11 gives the same lower bound of 21.

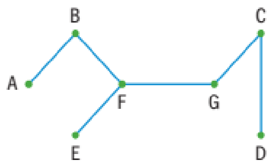
c For a Hamiltonian cycle on a complete graph, for each vertex we need two edges. The minimum weight of two of the edges adjacent to C is 9 and the minimum weight of two of the edges adjacent to E is 11. This adds up to 20. Five edges are needed for a cycle and as the smallest overall is 3 then the solution to the TSP must be at least 23.

d Start at A and move to the nearest vertex which has not already been visited but not B. When all other vertices have been visited go to B then to A. An alternative is to go to B first and then use the NNA as usual. The route would then be the reverse of the one obtained.

Chapter review

1 **a** There are exactly two odd vertices (C and F). **b** For instance, CDECBEFBAF

2 Select EF or DC. Then select CG, FB or AB, and GF. Total weight of the minimum spanning tree is 21.



3 There are two odd vertices A and D. The route of the lowest weight that connects them is AED ($40+38=78$). A possible route is ABCDEAEDFEBFCA with repeated edges AE and ED. The total weight of the graph is 394, and so the weight of the route is $394+78=472$.

4 **a** **i** The graph is not directed because it is symmetric.

ii The graph is not simple because it contains multiple edges.

iii The graph is not complete, because for example there is no edge between A and E.

iv The graph is not a tree because it contains a circuit, for example BCB.

b A degree 4, B degree 3, C degree 4, D degree 4, E degree 1.

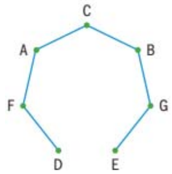
c **i** No because it has a vertex of degree 1.

ii No because not all vertices are even.

iii Yes, it has two odd vertices.

d $M+M^2=\begin{pmatrix}6&3&5&3&2\\3&5&3&4&0\\5&3&6&3&1\\3&4&3&6&1\\2&0&1&1&1\end{pmatrix}$, which has zero entries for walks of length 1 or length 2 between B and E. Hence, it is not possible to move between any pair of vertices in 1 or 2 steps.

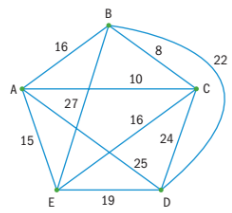
5 **a** Select edges AC, CB, BG, AF, FD, GE, which gives minimum spanning tree with the weight $10+8+9+11+13+14=65$.



b Remove the vertex H. Then the lower bound for the travelling salesman problem is given by $65+9+11=85$.

6 **a** **i** The graph is complete.

ii



iii Hamiltonian cycle is ACBDEA with weight $10+8+22+19+15=74$.

b **i** Remove A. Select edges BC, CE and ED. The minimum spanning tree of the subgraph has weight $8+16+19=43$.

ii A lower bound is $43+15+10=68$.

7 **a** **i** Yes, as the edges are directed. **ii** No, as it contains a loop.

iii Yes, as it contains a circuit that includes all the vertices.

b A has in-degree 2 and out-degree 2, B has in-degree 2 and out-degree 1, C has in-degree 1 and out-degree 1, D has in-degree 1 and out-degree 2.

c **i** The graph does not have Eulerian circuit.

ii For each vertex the in-degree must equal the out-degree.

iii Yes, DAABCDB.

iv All vertices must have in-degree equal to out-degree except for one vertex which has an out-degree one more than its in-degree and another vertex which has an in-degree one more than its out-degree.

d
$$\begin{array}{c} \\ \text{A} \\ \text{B} \\ \text{C} \\ \text{D} \end{array}\begin{array}{c} \begin{array}{cccc} \text{A} & \text{B} & \text{C} & \text{D} \end{array} \\ \begin{pmatrix} 0.5 & 0 & 0 & 0.5 \\ 0.5 & 0 & 0 & 0.5 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

e We solve $\begin{pmatrix} 0.5 & 0 & 0 & 0.5 \\ 0.5 & 0 & 0 & 0.5 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}=\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$, which gives a system of linear equations $-0.5x_1+0.5x_4=0$, $0.5x_1-x_2+0.5x_4=0$, $x_2-x_3=0$, and $x_3-x_4=0$. The probabilities $x_1+x_2+x_3+x_4=1$. This gives $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}=\begin{pmatrix} 0.25 \\ 0.25 \\ 0.25 \\ 0.25 \end{pmatrix}$. In a random walk approximately equal amounts of time will be spent at each vertex.

8 **a** C, D, F, G

b The possible pairings of these odd vertices are CD (1) and FEG (4), CEF (5) and DG (4), CBG (5) and DEF (4). The lowest weight pairing is CD and FEG, so the repeated edges will be CD, FE and EG. Possible walk is ABCDCEDGEFEGBFA with weight $4+3+1+1+3+2+4+2+2+2+2+2+3+1=32$.

c The total weight of the graph is 27. We want to begin at F (G) and end at G (F), so that we repeat CD, which is the pair with the lowest weight. Then the weight of the walk is $27+1=28$.

9 a The cycle would have to pass through vertex C twice to return to the starting point.

b For example ABCDE.

c

	A	**B**	**C**	**D**	**E**
A	0	13	5	12	9
B	13	0	8	15	12
C	5	8	0	7	4
D	12	15	7	0	10
E	9	12	4	10	0

d ACEDBA if mapped onto the original graph the route would be ACEDCBCA.

e 3

Exam style questions

10 a Every vertex is of even degree

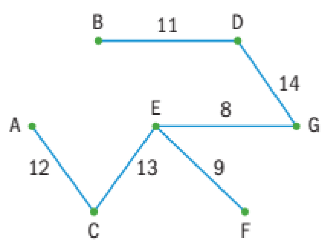
b One possibility is ACEFDEBDCBA

11 AF, AB, BC, FD, DE

$2 + 3 + 3 + 5 + 4 = 17$

12 a EG, EF, BD, AC, CE, DG

b



$11 + 14 + 8 + 9 + 13 + 12 = 67$
So cost is £6700

13 a FEDCABF

b For an Eulerian circuit, all vertices must be of even degree but there are two vertices of odd degree (B and E)

c BF, EF, AB or AC

This would then leave a vertex of degree 1, so no Hamiltonian cycle would be possible.

14 **a** C, D, E, F

b Consider the three pairings:
CD (3) and EF (5)

CE (7) and DF (8)

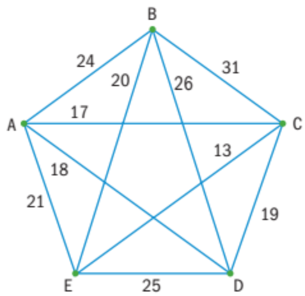
CF (3) and DE (4)

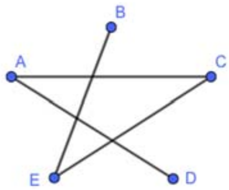
So shortest pairing is CF and DE, and repeated edges are CA, AF and DE.

Weight of route is therefore $41+(1+2+4)=48$

One possible route is AFACABFEBCDEDA

15 a





b CE, AC, AD, BE

c $13+17+18+20=68$

16 a $M=\begin{pmatrix}0&1&0&0\\1&0&1&1\\0&1&0&0\\0&0&1&0\end{pmatrix}$

b i $M^3=\begin{pmatrix}0&2&1&0\\2&1&2&2\\0&2&1&0\\1&0&1&1\end{pmatrix}$

ii 1

c i $S_3=M+M^2+M^3=\begin{pmatrix}0&1&0&0\\1&0&1&1\\0&1&0&0\\0&0&1&0\end{pmatrix}+\begin{pmatrix}1&0&1&1\\0&2&1&0\\1&0&1&1\\0&1&0&0\end{pmatrix}+\begin{pmatrix}0&2&1&0\\2&1&2&2\\0&2&1&0\\1&0&1&1\end{pmatrix}$

$=\begin{pmatrix}1&3&2&1\\3&3&4&3\\1&3&2&1\\1&1&2&1\end{pmatrix}$

ii 4

iii $B\to C$

$B\to D\to C$

$B\to A\to B\to C$

$B\to C\to B\to C$

17 a $T = \begin{pmatrix} 0 & \frac{1}{3} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 1 & 0 \\ 0 & \frac{1}{3} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3} & 0 & 0 \end{pmatrix}$

b $T^3 = \begin{pmatrix} \frac{1}{12} & \frac{1}{4} & \frac{1}{6} & \frac{3}{8} \\ \frac{5}{8} & \frac{1}{4} & \frac{1}{2} & 0 \\ \frac{1}{12} & \frac{1}{4} & \frac{1}{6} & \frac{3}{8} \\ \frac{5}{24} & \frac{1}{4} & \frac{1}{6} & \frac{1}{4} \end{pmatrix}$

The probabilities in the second column are all equal, so a ship starting from B is just as likely to be anywhere after three journeys.

c $\begin{pmatrix} 0.222 \\ 0.333 \\ 0.222 \\ 0.222 \end{pmatrix}$

d After a large amount of time, a ship is more likely to be based at port B, hence it would likely make the best place for the company's headquarters.

18 a

	A	**B**	**C**	**D**	**E**	**F**	**G**
A	0	4	5	10	9	9	3
B	4	0	3	8	5	5	5
C	5	3	0	5	8	8	2
D	10	8	5	0	6	13	7
E	9	5	8	6	0	7	10
F	9	5	8	13	7	0	10
G	3	5	2	7	10	10	0

b Best upper bound given by AGCBFEDA.
Weight is 36

c Actual route is AGCBFEDCG

19 a Transition matrix is $\begin{pmatrix} 0 & 0 & 0 & \frac{1}{3} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{3} & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \end{pmatrix}$

b Steady state vector is $\begin{pmatrix} 0.069 \\ 0.31 \\ 0.22 \\ 0.21 \\ 0.19 \end{pmatrix}$

So time spent at A is $0.069 \times 60 = 4$ minutes

time spent at B is $0.31 \times 60 = 19$ minutes

time spent at C is $0.22 \times 60 = 13$ minutes

time spent at D is $0.21 \times 60 = 13$ minutes

time spent at E is $0.19 \times 60 = 11$ minutes

20 a $M = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \end{pmatrix}$

b **i** $M^7 = \begin{pmatrix} 6 & 8 & 7 & 3 & 2 \\ 5 & 4 & 3 & 2 & 2 \\ 5 & 9 & 5 & 5 & 1 \\ 5 & 6 & 3 & 3 & 3 \\ 9 & 8 & 8 & 3 & 3 \end{pmatrix}$

Therefore the required journey is from C to E since there is only one element of 1.

ii CBABACDE

21 a Odd vertices are A, B, D, F
Consider three pairings:
AB DF has shortest route AB and DF, so repeated edges $6 + 5 = 11$
AD BF has shortest route AD and BC, CD, DF , so repeated edges $5 + 1 + 2 + 5 = 13$
AF BD has shortest route AD, DF and BC, CD , so repeated edges $5 + 5 + 1 + 2 = 13$
So best pairing is AB, DF

Nasson will need to travel along AB and DF twice.
Weight of route is therefore $53 + 6 + 5 = 64$

b One possible route is ACBABDCGFDFEDA

c If Nasson starts at F, the only possible routes that need to be repeated will either be AB (6), BD (BC+CD = 3) or AD (5).

BD is the shortest, so this should be repeated.

Therefore, given Nasson starts at F, he should finish at A.

22 a Order is ABCDEA.

Upper bound is $15 + 13 + 14 + 18 + 21 = 81$

b By deleting A, Kruskal gives MST for the remainder as BC, CD, CE; weight 43: Lower bound is therefore $43 + (15 + 17) = 75$

c By deleting B, Kruskal gives MST for the remainder as CD, CE, CA; weight 47; Lower bound is therefore $47 + (13 + 15) = 75$

d $75 \le L \le 81$

e Eg. tour for original upper bound: ABCDEA